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Kumamoto Journal of Science

Series A (Mathematics, Physics and Chemistry)

Vol. 2, No. 1, December 1954

Published by the
Faculty of Science, Kumamoto University
Kumamoto, Japan

熊本大學理學部紀要 第一部 第二卷 第一號 昭和二十九年十二月
熊 本 大 學 理 學 部

KATSUMASA
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昭和29年12月15日印刷
昭和29年12月20日發行　非賣品

編輯兼発行者　熊本大學理學部
印刷者　佐伯善次
印刷所　佐伯印刷所
熊本市大江町九品寺138

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昭和30年7月30日印刷
昭和30年7月30日發行
非賣品

編輯兼發行者　熊本大學理學部
印刷者　佐伯善次
印刷所　佐伯印刷所
熊本市大江町九品寺138

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REMARKS ON SPACES WITH NON-POSITIVE CURVATURE.

Yasuo NASU

(Received Sept. 30 1954)

H. Busemann [1], [2], [3] had dealt with the metric spaces called **G**-spaces or **E**-spaces by him. In his work [3], he defined **G**-spaces with non-positive curvature. His definition was a metric condition. But in differentiable Riemannian manifolds of class C^r $(r \geqq 4)$, this is equivalent to the condition of non-positive curvature in the usual sense. Among his works, it is to be in particular noticed that he systematically studied the relation between a ray and its coray and the theory of asymptotes. He [2], [3] obtained many interesting results in **G**-spaces or in **E**-spaces. On the other hand, F. P. Pederson [4] defined **G**-spaces with non-positive curvature under the more weak condition than the H. Busemann's and showed that H. Busemann's results hold again in this case. The purpose of this note is to show that F. P. Pederson's definition can be replaced by the more convenient form, which will be stated in §1.

1. ***Basic concepts and some theorems.*** To define an **E**-space (or a **G**-space), H. Busemann considered the following axioms: **A.** $\mathfrak{G}$ is a usual metric space with distance ab for any two points $a, b \in \mathfrak{G}$. **B.** $\mathfrak{G}$ is finitely compact. **C.** $\mathfrak{G}$ is convex metric. **D.** Every point x has a neighborhood $S(x, \alpha_x) = \{ y \mid xy < \alpha_x, yx < \alpha_x \} (\alpha_x > 0)$ such that for any positive number ε and any two points $a, b \in S(x, \alpha_x)$, there exists real numbers $0 < \delta_k(a, b) \leqq \varepsilon \ (k = 1, 2)$ for which the points a_1, b_1 satisfying the conditions $a_1 a + ab = a_1 b$, $aa_1 = \delta_1$; $ab + bb_1 = ab_1$, $bb_1 = \delta_2$ are uniquely determined.

When the above four axioms **A**, **B**, **C**, **D** are satisfied, $\mathfrak{G}$ is said an **E**-psace. Moreover, if the metric is symmetric, $\mathfrak{G}$ is said a **G**-space.

Following the usual method, we can define a continuous curve in an **E**-space (or in a **G**-space) and its length. For any two points p, q there exists a segment from p to q (or from q to p) and its length is equal to the distance pq (or qp). This fact easily follows from the axioms **A**, **B**, **C**. Under the axiom **D**, the prolongation of segments is locally possible and unique. We come from these facts to the concept of extremals.

G. Birkhoff [5] also gave another definition of **G**-spaces, which is apparently different from, but is equivalent to the H. Busemann's.

At first we shall formulate the following definition:

(1.1). ***Definition.*** *A continuous funtion $f(t)$ will be called semi-convex in the interval $[\alpha, \beta]$, if $f(t) \leqq max(f(t_1), f(t_2))$ for any t_1, t_2 with $\alpha \leqq t_1 < t < t_2 \leqq \beta$. If*

the equality never hold, then we shall call $f(t)$ strictly semi-convex.

In addition to this, if we assume that whenever $f(t) = max\,(f(t_1), f(t_2))$ for a value t with $t_1 < t < t_2$, then $f(t) = const.$ in $[t_1, t_2]$, then this definition coincides with F. P. Pederson's one.

The following properties of semi-convex functions are obvious from the definition.

(1.2) *If a continuons fuaction $f(t)$ is semi-convex in $[[\alpha, \beta]$, and has the minimum at the left or right end point of $[\alpha, \beta]$, then $f(t)$ is monotone. If $f(t)$ is a minimum at an inner point t_0 of $[\alpha, \beta]$, then $f(t)$ is monotone decreasing in $[\alpha, t_0]$ and monotone increasing in $[t_0, \beta]$. If $f(t)$ is strictly semi-convex, then $f(t)$ has the minimum value at only one point t_0. According as $t_0 = \alpha$, $\alpha < t_0 < \beta$, or $t_0 = \beta$, $f(t)$ is strictly increasing, strictly decreasing in $[\alpha, t_0]$ and strictly increasing in $[t_0, \beta]$, or strictly decreasing.*

H. Busemann showed that there exists a parametric representation $x(t)$, $-\infty < t < +\infty$, for an extremal $\mathfrak{x}$ such that

$\mathfrak{x}$: $x(t_1)x(t_2) = |t_2 - t_1|$ for sufficiently near t_1, t_2, where t denotes the arc-length of $\mathfrak{x}$. By making use of this, we shall define **G**-spaces with non-positive curvature.

(1.3) ***Definition.*** *In a **G**-space $\mathfrak{G}$, let $x(t)$, $\alpha_1 \leqq t \leqq \alpha_2$ $(\alpha_1 < \alpha_2)$; $y(s)$, $\beta_1 \leqq s \leqq \beta_2$ $(\beta_1 < \beta_2)$, be parametric representations of two extremal subarcs. Under the correspondence $s = ct + d$ $(c \neq 0)$ between t and s such that $\beta_i = c\alpha_i + d$ $(i = 1, 2)$, if the segment $T(x(t), y(ct+d))$ is unique for every t in $[\alpha_1, \alpha_2]$, and $f(t) = x(t)\,y(ct+d)$ is (strictly) semi-convex, then we say that $\mathfrak{G}$ has (negative) non-positive curvature.*

Let $\bar{\alpha}_x$ be the least upper bound of those α for which the segment $T(a, b)$ with end points a, b in $S(x, \alpha)$ is unique. H. Busemann's definition of **G**-spaces with non-positive curvature is as follows: The **G**-space is said to have non-positive curvature if every point x of $\mathfrak{G}$ has a nieghborhood $S(x, \beta_x)$ $(0 < \beta_x \leqq \bar{\alpha}_x)$ such that three points a, b, c in $S(x, \beta_x)$ satisfy the relation $2m(a, b)\,m(a, c) \leqq bc$, where $m(a, c)$, $m(b, c)$ are the middle points of a, b; a, c respectively. If, under the same conditions, $2m(a, b)\,m(a, c) = bc$, then $\mathfrak{G}$ is said to have curvature 0.

In the definiton (1.3), if $f(t)$ is convex, then $\mathfrak{G}$ has non-positive curature in H. Busemann's sene and vice versa [3]. The **G**-space $\mathfrak{G}$ with non-positive curvature in the present sense locally satisfies the condition (1.3), but the converse is not necessarily true. This easily follows from the fact that even if $f(t)$ is locally semi-convex, it is not necessarily semi-convex. From this, we can see that the condition of non-positive curvature in the present sense is weaker than H. Busemann's one.

The reason why we adopt the definition (1.3) is due to the following facts (i) and (ii).

(i) *F. P. Pederson defined **G**-spaces with (negative) non-positive curvature as follows: if every point x of a **G**-space has a neigehborhood $S(x, \gamma_x)$, $0 < \gamma_x \leqq \alpha_x$, such that for any*

two (non-collinear) segments T_1 *and* T_2 *in* $S(x, r_x)$, *the distances* $x(t)\,T_2$ *and* $y(t)\,T_1$ *are (strictly) semi-convex, where* $x(t)$, $\alpha_1 \leqq t \leqq \alpha_2$; $y(t)$, $\beta_1 \leqq t \leqq \beta_2$, *are respectively parametric representations of* T_1, T_2.

But the foot of $x(t)$ *on* T_2 *is not necessarily continuous even if* t *varies continuously. Therefore it seems difficult to derive (1.3) as theorem from F. P. Pedersonos's definition. It is important that, when* $\mathfrak{G}$ *is straight, F. P. Pederson's definition implies (1.3), as we can see from the results of H. Busemann [3].*

(ii) *The condition (1.1) of semi-convex functions is weaker than the F. P. Pederson's.*

The definition of non-positive curvature in the present sense coincides with the usual one in differentiable Riemannian manifolds in spite of such weak condition. Next we shall prove this.

(1.4). A Riemannian manifold V_n *of class* C^r $(r \geqq 4)$ *has non-positive curvatre in the usual sense if it has non-psitive curvature in the present sense.*

H. Busemann [3] proved that V_n has non-positive curvature in his sense if and only if it has non-positive curvature in the usual sense. Hence, if V_n has non-positive curvature in the usual sense, then $\mathfrak{G}$ has, of course, non-positive curvature in the present sense. Accordingly we shall prove that if V_n has the positive curvature $K(p, \tau)$ where $p \in V_n$ and τ is the surface formed by all geodesics tangent to a 2-plane element through p, then V_n hasn't non-positive curvature in the present sense.

For this purpose we shall prove

(1.5) If V_n *has the positive curvature* $K(p, \tau)$ *at* p $(\in V_n)$ *for a geodesic surface* τ *through* p *as above, then there is a neighborhood* $S(p, \beta_p)$ $(\beta_p > 0)$ *such that for any two points* $a, b \in S(p, \beta_p) \cap \tau$, *if* p, a, b *are non-collinear, the inequality*

$$ab < 2m(p, a)\, m(p, b) \tag{1.6}$$

holds.

From the well known theorem in differential geometry (see *e.g.* [6]), p has a neighborhood $S(p, \beta_p)$ such that for any two points $a, b \in S(p, \beta_p) \cap \tau$, if p, a, b are non-collinear, we have the following inequality

$$\gamma^2 < \alpha^2 + \beta^2 - 2\alpha\beta \cos \widehat{apb}, \tag{1.7}$$

where α, β, γ denote the geodesic distances pa, pb, ab respectively and $\widehat{apb}$ denotes the angle between the geodesic arcs $T(p, a)$, $T(p, b)$ at p.

When β_p is sufficiently small, we can assume that the Riemannain curvatures $K(m(p, b), \tau[T(m(p, b), p), T(m(p, b), m(p, a))])$, $K(m(p, b), \tau[T(m(p, b), m(p, a)), T(m(p, b), b)])$, $K(m(p, a), \tau[T(m(p, a), p)T, (m(p, a), b)])$, $K(m(p, a), \tau[T(m(p, a), a), T(m(p, a), b)])$ are positive, where $\tau[T(m(p, b), p), T(m(p, b), m(p, a))]$ is the

geodesic surface spanned by the two geodesic arcs $\boldsymbol{T}(m(p,b),p)$ and $\boldsymbol{T}(m(p,b),m(p,a))$ at $m(p,b)$, etc..

Putting $\alpha=2\lambda, \beta=2\mu, bm(p,a)=\nu, m(p,a)m(p,b)=\gamma'$, we have by (1,7)

$$\lambda^2<\gamma'^2+\mu^2-2\gamma'\mu\ cosm(p,a)\widehat{m(p,b)p},$$
$$\nu^2<\gamma'^2+\mu^2-2\gamma'\mu\ cosm(p,a)\widehat{m(p,b)b}.$$

Hence, we have by adding the above two inequalities

$$\lambda^2+\nu^2<2(\gamma'^2+\mu^2). \tag{1.8}$$

Similarly the inequality $\gamma^2+\beta^2<2(\lambda^2+\nu^2)$ holds. *I. e.*

$$\gamma^2+\beta^2<4(\gamma'^2+\mu^2).$$

Therefore we have $\gamma<2\gamma'$, since $\beta=2\mu$, which proves (1.5).

Proof of (1.4). Let $\mathbf{V}_n$ has the positive curvatnre $\boldsymbol{K}(p,\tau)$ for a geodesic surface τ through p. Consider a triangle pab where $a, b\in S(p,\beta_p/8)\cap\tau$ and the point e satisfying the conditions $m(p,a)\,e=2m(p,b)\,e=2m(p,a)\,m(p,b)$. Such a point e is uniquely determined because of

$$pm(p,a)<\beta_p/16,\quad pm(p,b)<\beta_p/16,\quad m(p,a)\,m(p,b)<\beta_p/8 \text{ and}$$
$$pe\leqq pm(p,b)+m(p,b)e<\beta_p/4.$$

Let d be a point such that $2be=2ed=bd$. Then

$$de=be\leqq pe+pb<\beta_p/2,\quad i.\ e.$$
$$pd\leqq pe+ed<\beta_p.$$

Thus such a point d is also uniquely determined. On the other hand, we may assume that $\boldsymbol{S}(p,\beta_p)$ is convex [7]. Therefore $\boldsymbol{T}(p,a)$, $\boldsymbol{T}(a,b)$, $\boldsymbol{T}(b,e)$, $\boldsymbol{T}(p,d)$ are contained in $\boldsymbol{S}(p,\beta_p)$. We then have $2m(p,b)\,e>pd$, since $\boldsymbol{K}(b,\tau[\boldsymbol{T}(b,p),\boldsymbol{T}(b,d)])$ is positve for sufficiently small β_p. From this inequality and (1.6) we conclude $m(p,a)e<pd$, $m(p,a)e>ab$. Thus we have completely proved (1.4).

The following theorem is clear.

(1.9) In a **G***-space* $\mathfrak{G}$ *with non-positive curvature, if the segments connecting a point* p *to points on an extremal subarc* $x(t), t_1\leqq t\leqq t_2$, *are unique then* $f(t)=px(t)$ *is semi-convex in* $[t_1, t_2]$.

It is to be noticed that $f(t)$ can not necessarily be strictly semi-convex, even if $\mathfrak{G}$ has negative curvature. (1.9) yields

(1.10) $\boldsymbol{S}(p,\bar{\alpha}_p)$ *is convex.*

We shall say that $\mathfrak{G}$ has negative curvature in strong sense, if $\mathfrak{G}$ has negative curvature and is straight and the function $f(t)$ in (1.9) is strictly semi-convex whenever $p \in x(t)$, $t_1 \leqq t \leqq t_2$,.

(1.11) Let a G-space $\mathfrak{G}$ has non-positive curvature in H. Busemann's sense, and be straight. If a point p ($\in \mathfrak{G}$) has a foot f on a straight line $\mathfrak{l}$ ($\ni p$), then every point on the half ray $\mathfrak{k}$ which has the initial point f and contains p has only one foot f on $\mathfrak{k}$.

Proof. Let $x(t), -\infty < x < +\infty$, be a parametric representation of $\mathfrak{l}$. Then $f(t)=px(t)$ is strictly convex [3]. Hence p has only one foot f on $\mathfrak{l}$. On the other hand, it is easily proved that every point on the subarc $T(f, p)$ has a unique foot f on $\mathfrak{l}$ except p. Consider now a point q on $\mathfrak{k}$ such that $q \in T(p, f)$ and $q\, m(p, f) < pf$. Let f' be the foot of q on $\mathfrak{l}$. We assume $f \neq f'$. Then $qf > qf'$ because $qx(t)=g(t)$ is strictly convex. From the assumption of (1.11), we have

$$m(q, f)m(f, f') \leqq \frac{1}{2} qf' < \frac{1}{2} qf$$

Therefore

$$m(q, f)m(f, f') < m(q, f)f.$$

This fact shows us that the foot of $m(q, f)$ ($\in T(f, q)$) is different from f, which contradicts the above mentioned fact. This proves (1.11).

It seems difficult to prove (1.11) in G-spaces with non-positve curvature in the present sense. But we can prove the following theorem.

(1.12) If a G-space $\mathfrak{G}$ has negative curvature in strong sense, then a point p ($\in \mathfrak{G}$) has only one foot on a straight line $\mathfrak{l}$ ($\ni p$), and when p varies continuously, the foot also varies continuously.

Proof. Let $x(t), -\infty < t < +\infty$, be a parametric representation of $\mathfrak{l}$. Then $px(t) = g(t)$ is strictly semi-convex. Accordingly p has only one foot f on $\mathfrak{l}$. Let $\{p_n\}$ be a sequence of points which converges to a point p and f_n be the foot of p_n on $\mathfrak{l}$ $(n=1, 2)$. It is sufficient to prove that $\{f_n\}$ converges to the point f which is the foot of p. If it were not the case, there would be a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k} \to f'$ as $n_k \to \infty$ and $f' \neq f$. Then $T(p_{n_k}, f_{n_k})$ converges uniformly to $T(p, f)$ and we can conclude that f' is the foot of p, which contradicts that f is the unique foot of p. Therefore $\{f_n\}$ converges to f.

2. ***The universal covering spaces for a G-space with non-positive curvature.*** In this paragraph, we assume that G-spaces have the property of domain invariance. This assumption was adopted by H. Busemann [3], and formulated as follows: "If X is a subset of a G-space $\mathfrak{G}$ and is homeomorphic to an open set U, then X is open in $\mathfrak{G}$".

By making use of this assumption, he extended the well known theorem in Riemannian spaces to that of G-spaces, that is to say,

(2.1) If a G-space $\mathfrak{G}$ satisfies the property of domain invariance and has non-positive curvature then the universal covering spaces for $\mathfrak{G}$ are straight, and have non-positive curvature.

In the above theorem, if $\mathfrak{G}$ has non-positive curvature in H. Busemann's sense and satisfies the property of domain invariance, then the universal covering spaces for $\mathfrak{G}$ also have the same properties and are straight [3]. Next we shall prove (2.1) in the present sense.

Proof. Fix a point $p\,(\in \mathfrak{G})$, and put $K=\{x \mid px=\alpha_p/2\}\,(\alpha_p>0)$. For $u\in K$ let $x(u,t)$, $0\leqq t<+\infty$, be the parametric representation of the half extremal issuing from the point $p\,(=x(u,0))$ and containing u. Then (u,t) and $x(u,t)$ are in one to one correspondence for $0\leqq t<\alpha_p$. Fix u_0 and $t_0>0$.
Then

$$\delta = \inf_{0\leqq t\leqq t_0} \alpha_{x(u_0,t)}$$

is positive. Consider a sequence $\{u_n\}\,(\subset K)$ which converges to u_0. $\{x(u_n,t)\}$ converges uniformly to $x(u_0,t)$ for all t in $[0,t_0]$. Therefore there exists a positive integer N such that

$$\bigcup_{0\leqq t\leqq t_0} S(x(u_0,t),\ \delta/4) \supset x(u_n,t),\ \ 0\leqq t\leqq t_0,\ \ \textit{for } n\geqq N.$$

We shall now prove that each extremal subarc of $\{x(u_n,t),\ 0\leqq t\leqq t_0\}\,(n\geqq N)$ disjoins the others except p. For, let $x(u_{n_0},t),\ 0<t\leqq t_0$, intersects $x(u_0,t),\ 0\leqq t\leqq t_0$, at $x(u_0,t')\ (t'\neq 0)$. Then $x(u_{n_0},t)\,x(u_0,t)<\delta/4$. Let t' be the first value of t for which $x(u_0,t),\ 0<t\leqq t_0$, intersects $x(u_{n_0},t),\ 0<t\leqq t_0$. Put $x(u_0,t')=x(u_{n_0},t'')$ and $\bar{t}'=\bar{t}\,t''/t'\ (0\leqq\bar{t}\leqq t')$.
Then

$$|\bar{t}'-\bar{t}\,| = |t''-t'\,|\,\bar{t}/t'\leqq |t''-t'\,| = x(u_0,t')\,x(u_0,t'')<\delta/4.$$

we have from this

$$x(u_{n_0},\bar{t}')x(u_0,\bar{t})\leqq x(u_{n_0},t')x(u_0,\bar{t}')+x(u_0,\bar{t}')x(u_0,\bar{t})<\delta/2.$$

Therefore the $T(x(u_{n_0},t'),\ x(u,t))$ is unique for every pair $(t,\bar{t}')$, and the functon $f(t)=x(u_{n_0},\ \bar{t}\,t''/t')\,x(u_0,\ \bar{t})$ is semi-convex, but $f(t')=f(0)=0$, which contradicts the definition of the semi-convexity. Similarly we can prove that every $x(u_n,t),\ 0<t\leqq t_0\,(n\geqq N)$ hasn't common points with the others.

On the other hand, we can find that all half extremals $x(u,t)$, $0 \leqq t \leqq \alpha_p$, $(u \in S(u_0, \delta/4) \cup K)$ simply cover a neighborhood $S(u_0, \varepsilon)$ $(0 < \varepsilon < \delta/4)$. For if this were not the case, then there would be a sequence $\{v_n\}$ which converges to u_0 and is not contained in $\{x(u,t) \mid 0 \leqq t \leqq \alpha_p,\ u \in S(u_0, \delta/4) \cap K\}$. Every half extremal passing through v_n and having p as initial point intersects in the first K at only one point u_n, then $\{u_n\}$ converges to u_0 as $\{T(p, v_n)\}$ converges to $T(p, u_0)$. Therefore there exists a positive integer N such that $S(u_0, \delta/4) \cap K \ni u_n$ for $n \geqq N$, which contradicts our assumption.

The half extremal connecting p to $q(\in S(u_0, \varepsilon))$ is unique and expressible by $x(u', t)$, $0 \leqq t < +\infty$, Put $q = x(u', t)$. The point $x(u', t')$ corresponding by the relation $t' = t_0 - \varepsilon/2 + t$ to q makes a set of points U as q varies in $S(u_0, \varepsilon)$. This correspondence is one-to-one and bicontinuous. Therefore U is open by the assumption of domain invariance. We know in the above correspondence that $U \ni x(u_0, t_0)$. Hence we conclude

(2.2) If a G-space $\mathfrak{G}$ has non-positive curvature and satisfies the property of domain invariance, then every point of a half extremal $\mathfrak{x}$ has a neighborhood simply covered by a system of half extremals containing $\mathfrak{x}$ and having the same initial point as $\mathfrak{x}$. All points in such a neighborhood are locally in one-to-one correspondence with pairs (u, t).

Let $\mathfrak{G}$ be the set of pairs (u, t), i. e., the topological product of K and half real line: $0 \leqq t < +\infty$. Then every element $(u, t)(\in \mathfrak{G})$ has a neighborhood W which is in one-to-one correspondence with a sufficiently small neighborhood U of $x(u, t)$. Hence we can define locally and isometrically a metric on $\mathfrak{G}$ by making use of the above correspondence between W and U. Therefore we can define a homeomorphism ϕ: $W \to U$. We can extend the definition of this homeomorphism on all $\mathfrak{G}$. Thus we have a homomrphism ψ: $\mathfrak{G} \to \mathfrak{G}$. Hence $\mathfrak{G}$ is a covering space for $\mathfrak{G}$. We can define a curve length of any continuous curve connecting two elements (points) in $\mathfrak{G}$. From this the distance of any two points is defined by the least upper bound of lengths of all continuous curves connecting these two points. Then any two points (u_0, t), (v_0, s_0) can be connected by the arcs (u_0, t), $0 \leqq t \leqq t_0$; (v_0, s), $0 \leqq s \leqq s_0$. Hence the distance $(u_0, t_0)(v_0, s_0)$ is finite and symmetric. Hence the axiom A is satsfied in $\mathfrak{G}$. Since $\mathfrak{G}$ is a covering space for $\mathfrak{G}$ and $(u,t)(v,s) \geqq x(u,t)\,x(v,s)$ for any two points of $\mathfrak{G}$, if a set of points $\{(u_\nu, t_\nu)\}$ is bounded, then $\{q_\nu\}$ $(q_\nu = x(u_\nu, t_\nu))$ is bounded in $\bar{\mathfrak{G}}$. Therefore $\{q_\nu\}$ has an accumulation point q. From this, we can easily prove that the axiom B is satisfied in $\mathfrak{G}$. It is also clear that the axioms C, D are satisfied in $\bar{\mathfrak{G}}$. Hence $\bar{\mathfrak{G}}$ is a G-space.

We shall next prove that $\mathfrak{G}$ is simply connected. If we consider a simple continuous curve $\mathfrak{C}$ in $\mathfrak{G}$, then we can establish a continuous correspondence between the pairs (u, t) and the points of $\mathfrak{G}$. At this time, $\mathfrak{G}$ is contractible to p. Hence $\mathfrak{G}$ is simply connected and is a universal covering space for $\mathfrak{G}$.

We shall next prove that $\overline{\mathfrak{G}}$ is straight. To do this, we shall show that for any fixed u the curve (u, t), $0 \leqq t < \infty$, is a ray or that the arc (u, t), $0 \leqq t \leqq t_0$, is a shortest connection of $\bar{p}=(u, 0)$ and (u, t_0) for every t_0, because t_0 is the arc length of (u, t), $0 \leqq t \leqq t_0$, from the defintion. Let α be the least upper bound of those t_0. Then $\alpha \geqq \alpha_p > 0$. Therefore we have to show $\alpha = +\infty$. If α is finte, then there exists a half extremal $x(u, t)$, $0 \leqq t < +\infty$, such that (u, t), $0 \leqq t \leqq \alpha+\rho$, $(\rho > 0)$ is not segment. But in this case a segment connecting $(u, \alpha+\rho)$ to $p(u, 0)$ can be represented by (v, t), $0 \leqq t \leqq$ the distance $(u, 0)(u, \alpha+\rho)$, which contradicts the definition of $\mathfrak{G}$.

From the fact mentioned above, every half extremal issuing from the point $\bar{p}$ is a ray in $\mathfrak{G}$. Let $\bar{q}$ $(=(u_0, t_0))$ be a point of $\mathfrak{G}$ different from $\bar{p}$. Because $\mathfrak{G}$ has non-positive curvature and $\mathfrak{G}$ is a universal covering space for $\mathfrak{G}$, every point of a half extremal with the initial point $\bar{q}$ has a neighborhood simply covered by a system of half extremals issuing from the point $\bar{q}$. Let two half extremals $\mathfrak{x}, \mathfrak{x}'$ having the same initial point $\bar{q}$ interect at $\bar{r}$. At this time, we may assume $\bar{r} \neq \bar{p}$. Then $\bar{r}$ has a neighborhood U simply covered by the two systems of half extremals passing through neighborhoods of $\mathfrak{x}, \mathfrak{x}'$. Consider a continus curve $\mathfrak{C}'$ connecting $\bar{p}$ to $\bar{r}$, we can from these systems pick up two half extremals, which intersect at a point $\bar{r}'$ in $U \cap \mathfrak{C}'$. We can apply the same consideration to $\bar{r}'$. Thus when $\bar{r}'$ tends to p, these two half extremals don't coincide because $\mathfrak{G}$ has non-positive curvature and $\mathfrak{G}$ has the same local property as $\mathfrak{G}$. There exist through $\bar{p}$ two distinct extremals with a point q in common. Hence every half extremal with the initial point $\bar{q}$ is a ray. We see that $\overline{\mathfrak{G}}$ is straight.

Next, we shall prove the last part of (2.1).

Let $\bar{x}(t)$, $\alpha_1 \leqq t \leqq \alpha_2$; $\bar{y}(s)$, $\beta_1 \leqq s \leqq \beta_2$, be two extremal subarcs in $\overline{\mathfrak{G}}$. As we can see from the above proof, there exists a real number $\bar{\varepsilon}$ for the segment $\bar{x}(t)$, $\alpha_1 \leqq t \leqq \alpha_2$ such that if any extremal subarc $\bar{z}(t)$, $\gamma_1 \leqq t \leqq \gamma_2$, which is a segment in this case, is contained in $\bigcup_{\alpha_1 \leqq t \leqq \alpha_2} S(\bar{x}(t), \bar{\varepsilon})$ and $z(\gamma_i) \cdot S(\bar{x}(\alpha_i), \bar{\varepsilon})$ $(i=1, 2)$, then the function $f(t) = \bar{x}(t)\,\bar{z}(at+b)$ is semi-convex, where $a\gamma_i+b=\alpha_i$ $(i=1, 2)$. If we put $\bar{\varepsilon}$ the lower bound of those ε for all segments connecting two points which divide the segments $T(\bar{x}(\alpha_i), y(\beta_i))$ $(i=1, 2)$ into same ratio, then $\varepsilon > 0$. We divide two segments $T(\bar{x}(\alpha_i), y(\beta_i))$ $(i=1, 2)$ into n parts of equal lengths d_i/n $(i=1, 2)$ by points $\bar{x}(\alpha_1)=\bar{x}_0^1, \bar{x}_1^1, \cdots\cdots, \bar{x}_1^{n-1}, x_1^n = \bar{y}(\beta_1)$ and by points $\bar{x}(\alpha_2)=\bar{x}_2^0, \bar{x}_2^1, \cdots\cdots, \bar{x}_2^{n-1}, \bar{x}_2^n = \bar{y}(\beta_2)$, where d_1, d_2 are respectively the arc-lengths of $T(\bar{x}(\alpha_1), y(\beta_1))$, $T(\bar{x}(\alpha_2), y(\beta_2))$. If n is sufficiently large, then the ε-neighborhood of the segment $T(\bar{x}_1^i, \bar{x}_2^i)$ contains the segments $T(\bar{x}_1^{i-1}, \bar{x}_2^{i-1})$, $T(\bar{x}_1^{i+1}, \bar{x}_2^{i+1})$ for every $i = 1, 2, \cdots\cdots, n-1$. Let $\bar{x}^0, \bar{x}^1, \cdots\cdots, \bar{x}^n$ be points dividing the segments $T(\bar{x}_1^i, \bar{x}_2^i)$ $(i=0, 1, 2, \cdots\cdots, n)$ into a ratio $t-\alpha_1 : \alpha_2-t$ $(\alpha < t$

$<\alpha_2)$ where $\bar{x}' = \bar{x}(t)$, $\bar{x}'' = \bar{y}(a\bar{t} + \bar{b})$ and $\alpha_i = \bar{a}\beta_i + \bar{b}\,(i = 1, 2)$. Then we have

$$\bar{x}(\bar{t})\,\bar{y}(\bar{a}\bar{t} + \bar{b}) \leqq \sum_{i=0}^{n-1} \bar{x}^i\,\bar{x}^{i+1} \leqq max\,(d_1, d_2).$$

The above proof is applicable to any subinterval of $[\alpha_1, \alpha_2]$. Hence we can see that if $\mathfrak{G}$ has (negative) non-positive curvature $\overline{\mathfrak{G}}$ also has (negative) non-positive curvature. Thus we complete the proof of (2.1).

(2.3) *If a simply connected* G*-space* $\mathfrak{G}$ *has non-positive curvature and satisfies the property of domain invariance, then* $\overline{\mathfrak{G}}$ *is straight.*

This theorem easily follows from (2.1). The following theorem is clear from the proof of (1.2).

(2.4) *In a* G*-space* $\mathfrak{G}$, *if, for any point* $p\,(\in \mathfrak{G})$ *and for any bounded extremal subarc issuing from* p, *a neighborhood of every point except* p *of this subarc is simply covered by a system of half extremals with the same initial point* p, *then the universal covering spaces for* $\mathfrak{G}$ *are straight.*

(2.5) *If a compact* G*-space* $\mathfrak{G}$ *has non-positive curvature and satisfies the property of domain invariance, then* $\mathfrak{G}$ *is not simply connected but has finite connectivity.*

This theorem is clear from (2.1). The last part also can be easily proved.

3. ***Asymptotes and some remarks.*** E. Cartan [6] had dealt with the relation between a ray and its coray in a hyperbolic space. The theory of asymptotes is also due to his work. Later H. Busemann systematically studied these objects. In an E-space $\mathfrak{E}$, let $\mathfrak{l}$ be a ray whose parametric representation is $x(t)$, $0 \leqq t < +\infty$, let $\{t_n\}$ be a sequence of real numbers such that $t_n \to +\infty$ as $n \to \infty$ and let $\{p_n\}$ be a Cauchy sequence which converges to a point p. Then the sequence of segments $\{T(p_n, x(t_n))\}$ contains a subsequence coverging to a ray $\mathfrak{x}$ and having the initial point p, and $\mathfrak{x}$ is said a coray from p to $\mathfrak{l}$. H. Busemann [2] proved that the above limit is equivalent to the closed limit introduced by Hausdorff.

It is clear that there is not necessarily a unique coray $\mathfrak{x}$ from p to $\mathfrak{l}$. But if $q \in \mathfrak{x} - p$, then there is one and only one coray from q to $\mathfrak{l}$, and the coray is a subray of $\mathfrak{x}$ [3]. The union $\mathfrak{A}$ of all the cariers of the corays to $\mathfrak{l}$ which contain $\mathfrak{x}$ as subray is said an asymptote to $\mathfrak{l}$. If the asymptote $\mathfrak{A}$ carries a straight line, then there does not exist the initial point of $\mathfrak{A}$ but generally exists. The initial point of $\mathfrak{A}$ is said an asymptotic conjugate point to $\mathfrak{l}$. There arise many interesting problems for the set of asymptotic conjugate points, and some of the results [8] obtained by the author will be published elsewhere.

Specialy, if $\mathfrak{G}$ is straight, then the set of asymptotic conjugate points to a ray $\mathfrak{l}$ is empty. In other words, every asymptote carries a straight line. In this case, the theory

of asymptotes is simple and the sequence $\{p_n\}$ in the defintion of corays can be substituted by only one point p. If a G-space $\mathfrak{G}$ has non-positive curvature in H. Busemann's sense and is straight, then the concept of asymptotes is symmetric and transitive, in other words, if $\mathfrak{A}$ is an asymptote to $\mathfrak{l}$, then the straight line carrying $\mathfrak{l}$ is an asymptote to $\mathfrak{A}$. H. Busemann [3] obtained the many results by making use of this property. But even if $\mathfrak{G}$ has non-positive curvature in the present sense and is straight, the concept of asymptotes is not necessarily symmetric and transitive. Therefore many results corresponding to the Busemann's cannot be obtained. We also come to the same conclusion in a straight G-space with non-positive curvature in the sense of F. P. Pederson. F. P. Pederson assumed in [4] the divergence property: "*Two rays* $\mathfrak{x}$ and $\mathfrak{x}'$ *having the same initial point satisfy the conditions:*

$$\lim_{t\to\infty} x(t)\,\mathfrak{x}' = \lim_{t\to\infty} x'(t')\,\mathfrak{x} = +\infty,$$

where $x(t)$, $0 \leqq t < +\infty$,; $x'(t')$, $0 \leqq t' < +\infty$, *denote parametric representations of* $\mathfrak{x}$, $\mathfrak{x}'$ *respectively*".

Under this assumption, he obtained many results analogous to the Busemann's. We can also in the present sense obtain these results. But it is important to investigate more precisely the properties of divergence and domaine invariance. We hope to study these two properties and moreover the property of set of the asymptotic conjugate points to a ray in differentiable Riemannian or Finsler manifolds in a later paper.

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ON THE NORMALITY IN MINKOWSKIAN SPACES

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(Received Sept. 30 1954)

In an n-dimensional Euclidean space $\mathbf{R}$, let a function $F(\xi)$ defined on vectors $X=(\xi^1, \xi^2, \cdots\cdots, \xi^n)$ be positive homogeneous of the first order in the variables ξ's. If C: $F(\xi)=1$ is convex, then we can define a Minkowskian space $\mathbf{R}$ by making use of C as unit sphere. Specially, if C is convex and differentiable, a transversal hyperplane to a vector $X_0=(\xi_0^1, \xi_0^2, \cdots\cdots, \xi_0^n)$ is given by

$$\sum_{i=1}^{n}\left(\frac{\partial F}{\partial \xi^i}\right)(\xi^i-\xi_i^0)=0. \tag{0.1}$$

In this case, we say that the vector X_0 is normal to the hyperplane (0,1) or that the hyperplane (0,1) is transversal to X_0. H. Busemann extended the concepts of the normality and the transversaltiy by making use of the Sm-function defined in his work [1] and he briefly proved A. E. Taylor's theorem (see [2]). On the other hand, W. Barthel [3] studied Minkowskian spaces from the standpoint of volume and area defined by H. Busemann. If C is merely strictly convex, we can introduce the concepts of curvature and curve length by using Minkowskian metric, area, and volume in place of Euclidean metric, area, and volume respectively. We shall devote this note to extend (0,1) to the conditon of normality and transversality in H. Busemann's sense. W. Barthel [3] incompletely studied this problem. We shall deal with the part overlooked by him. We assume in this note that C has a point as center, i. e., $F(\xi)=F(-\xi)$.

1. Let V be a set on an a-flat A in $\mathbf{R}$, then the Minkowskian area $|V|^M$ is defined by

$$|V|^M=\sigma(A)\,|V|^L,\quad \sigma(A)=\omega^{(a)}/|U(A)|^L, \tag{1,1}$$

where $\omega^{(a)}=\pi^{a/2}/\Gamma\left(\frac{a}{2}+1\right)$ is the volume of the Euclidean unit sphere in A, $|V|^L$ denotes the Lebesgue measure of V, and $U(A)$ denotes the point set in which the a-flat parallel to A through the center of C intersects the interior of C.

Let the a-flat A intersects the b-flat B in the d-flat D where d is given by $min(a,b)=d+1$. At this time, $sin(A,B)$ is defined and $b\,(=a+b-d)$ is the dimension of the flat Q spanned by A and B. We shall call m-box an m-dimensional parallelepiped, and denote it by P_m. Let P_d be a proper d-box in D and P_a, P_b proper a and b-boxes

in $\boldsymbol{A}$ and $\boldsymbol{B}$ respectively, which contain $\boldsymbol{P}_d$ as face. Let $\boldsymbol{P}_q$ be the q-box spanned by the boxes $\boldsymbol{P}_a$ and $\boldsymbol{P}_b$. Then we define the function $\boldsymbol{Sm}(\boldsymbol{A}, \boldsymbol{B})$ as follows:

$$(1,2) \qquad \begin{aligned} \boldsymbol{Sm}(\boldsymbol{A}, \boldsymbol{B}) &= |\boldsymbol{P}_d|^M |\boldsymbol{P}_q|^M / |\boldsymbol{P}_a|^M |\boldsymbol{P}_b|^M \\ &= sin\,(\boldsymbol{A}, \boldsymbol{B})\, \sigma(\boldsymbol{D})\, \sigma(\boldsymbol{Q}) / \sigma(\boldsymbol{A}) \sigma(\boldsymbol{B}). \end{aligned}$$

We can easily show that the above definition does not depend on the choice of the boxes $\boldsymbol{P}_a$, $\boldsymbol{P}_b$, $\boldsymbol{P}_d$. *We say that* $\boldsymbol{A}$ *is normal to* $\boldsymbol{B}$ *in* $\boldsymbol{Q}$ *at* $\boldsymbol{D}$, *or* $\boldsymbol{B}$ *is transversal to* $\boldsymbol{A}$ *in* $\boldsymbol{Q}$ *at* $\boldsymbol{D}$, *when* $\boldsymbol{Sm}(\boldsymbol{A}^*, \boldsymbol{B}) \leqq \boldsymbol{Sm}(\boldsymbol{A}, \boldsymbol{B})$ *for any a-flat* $\boldsymbol{A}^*$ *through* $\boldsymbol{D}$ *in* $\boldsymbol{Q}$.

If we take a system of orthogonal unit vectors $\boldsymbol{X}_i = (\xi_i^1, \xi_i^2, \cdots\cdots, \xi_i^n)\,(i=1,2,\cdots,a)$ parallel to $\boldsymbol{A}$, then $\sigma(\boldsymbol{A})$ is represented by

$$(1,3) \qquad \sigma(\boldsymbol{A}) = \omega^{(a)} \Big/ \int_{F(\xi_i u^i) \leqq 1} du^1 \cdots du^a$$

Hereafter we shall put $\sigma(\boldsymbol{A}) = \boldsymbol{F}^{(a)}(\boldsymbol{X}_1, \cdots\cdots, \boldsymbol{X}_a)$. Let $\boldsymbol{X}'_i\,(i=1,2,\cdots\cdots,a)$ be other independent vectors parallel to $\boldsymbol{A}$, then we have

$$\boldsymbol{X}'_k = \sum_i C_k^i \boldsymbol{X}_i, \quad det \parallel C_k^i \parallel \neq 0.$$

By putting $u^i = \sum_k C_k^i u^{k'}$, we have

$$(1,4) \qquad \begin{aligned} \boldsymbol{F}^{(a)}(\boldsymbol{X}'_1 \cdots, \boldsymbol{X}'_a) &= \omega^{(a)} \Big/ \int_{F(\xi'_i u^{i'}) \leqq 1} du^{1'} \cdots du^{a'} \\ &= \boldsymbol{F}^{(a)}(\boldsymbol{X}_1, \cdots\cdots, \boldsymbol{X}_a) \left| det \parallel C_k^i \parallel \right|. \end{aligned}$$

Hence we have the function $\boldsymbol{F}^{(a)}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots\cdots\cdots, \boldsymbol{X}_a)$ defined on linealy independent vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots\cdots, \boldsymbol{X}_a$. If $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots\cdots, \boldsymbol{X}_a$ are in partciular linearly dependent, then we assume $\boldsymbol{F}^{(a)}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots\cdots, \boldsymbol{X}_a) = 0$. Next, we shall deal with the relation between the differentiability of $\boldsymbol{F}^{(a)}(\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots\cdots, \boldsymbol{X}_a)$ with respect to the cempomnents of vectors and that of the indicatrix function $\boldsymbol{F}(\xi)$ with respect to the variables ξ^i s.

(1,5). Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots\cdots\cdots, \boldsymbol{X}_{a-1}$ *be a system of orthogonal unit vectors parallel to an* $(a-1)$*-flat* Φ *containing the center of* $\boldsymbol{C}$; *let* $\boldsymbol{C}_\Phi$ *be the intersection of the interior of* $\boldsymbol{C}$ *and* Φ; *let a vector* $\boldsymbol{X}_a$ *be perpendicular* to* Φ; *let* Ψ *be a half a-flat spanned by the vector* $\boldsymbol{X}_a$ *and* Φ; *and* Π *be a 2-flat perpendicular to* Ψ *at* O *which contains* $\boldsymbol{X}_a$. *If* $|\Psi \cap \boldsymbol{C}|^L$ *is laid off on the ray* $\Psi \cap \Pi$ *from* O *when* Ψ *varies under the above condition and* Φ, Π *are fixed, then the resulting curve* Γ *in* Π *is differentiable when* $\boldsymbol{F}(\xi)$ *is dfferentiable.*

* Hereafer we use the term "perpendicular" in the Euclidean sense.

Furthermore the order of differentiability of Γ is not less than that of $F(\xi)$. If C is (strictly) convex, then Γ is also (strictly) convex.

We introduce rectangular coordinates $(x, y, u^1, \cdots\cdots, u^{a-1})$ with O as origin, such that $//$ is the (x, y)-plane and Φ is the $(u^1, \cdots\cdots, u^{a-1})$-plane. We also introduce polar coordinates (γ, φ) in $//$. Then the equations of Γ are given by

$$x=\left(\int_{C_\Phi} r(\varphi; u^1, \cdots\cdots, u^{a-1})\, du^1 \cdots du^{a-1}\right) \cos\varphi,$$

$$y=\left(\int_{C_\Phi} r(\varphi; u^1, \cdots\cdots, u^{a-1})\, du^1 \cdots du^{a-1}\right) \sin\varphi,$$

where $r(\varphi; u^1, \cdots\cdots, u^{a-1})$ is the length of segment cut off by $C \cap \Psi$ on the ray perpendicular to C_Φ at $(o, o, u^1, \cdots\cdots, u^{a-1})$ and φ is the angle between the ray $// \cap \Psi$ and x-axis. If C is (strictly) convex, then the functions $\gamma(\varphi; u^1, \cdots\cdots, u^{a-1}) \cos\varphi$ and $\gamma(\varphi; u^1, \cdots\cdots, u^{a-1}) \sin\varphi$ are both (strictly) convex as φ varies and $u^1, \cdots\cdots, u^{a-1}$ are fixed. This fact yields that, if C is (strictly) convex, then Γ is also (strictly) convex.

On the other hand, $r(\varphi; u_0^1, \cdots\cdots, u_0^{a-1})$ is the moving radius of the intersection of C and the 2-flat parallel to $//$ at $(o, o, u_0^1, \cdots\cdots, u_0^{a-1})$ whose parameter is φ. If we take a rectangular coordinate system $(x, y, u^1, \cdots\cdots, u^{a-1}, u^a, \cdots\cdots u^{n-2})$ with O as origin, then $\xi^1, \cdots\cdots, \xi^n$ are represented by the linear combination:

$$\xi^i = a_1^i x + a_2^i y + \sum_{k=3}^{n-2} a_k^i u^k \quad (i=1, 2, \cdots\cdots, n).$$

In the equaton of C, if we substitute $(x, y, u^1, \cdots\cdots, u^{n-2})$ *for* $(\xi^1, \cdots\cdots, \xi^n)$ and put $x=\gamma\cos\varphi$, $y=\gamma\sin\varphi$, $u^1=u_0^1$, $u^2=u_0^2$, $\cdots\cdots$, $u^{a-1}=u_0^{a-1}$, $u^a=\cdots\cdots=u^{n-2}=0$, we then have the equation of the intersection of C and the 2-flat parallel to $//$ at $(o, o, u_0^1, \cdots\cdots, u_0^{a-1})$ as an implicit function.

From the theorem of implicit functions and the differentiability of $F(\xi)$, we see that the order of the differentiability of Γ with respect to φ is not less than that of $F(\xi)$ in the ξ's. Thus we complete the proof of (1.5).

We can easily see from the explanation in the next paragraph that $F^{(a)}(X_1, X_2, \cdots\cdots, X_a)$ is differentiable with respect to the components of vectors, when $F(\xi)$ is differentiable, and that the order of differentability is not less than that of $F(\xi)$.

2. In this paragraph, we shall directly extend (0.1) to the condition of normality and transversality in H. Busemann's sense. Generally that A is normal to B does not

imply that B is normal to A. Therefore we must distinguish the two cases $a=d+1$ and $b=d+1$ for our purpose.

I $a=d+1$. Let A^* be an a-flat perpendicular to B at D; *let* z be a vertex of P_d, and let zz^* be the ray perpendicular to B at z. Then zz^* is contained in A^*. Let B^* be the b-flat containing the b-face parallel to B. Then the a-box P_a is cut off by B and B^* on the a-flat A through D in Q, Let θ be the angle between the ray zz^* and the ray zx perpendicular to D in A. Then we have

$$|P_a|^{\mathrm{L}} = \sec\theta \, |P_a^*|^{\mathrm{L}},$$

where P_a^* is the a-box cut off by B and B^* on A^*. From the definition of $Sm(A, B)$, we can easily see that A is normal to B, when $\sec\theta\ \sigma(A)$ is minimal (or $U(A)\,|^{\mathrm{L}} \cos\theta$ is maximal). Let $(X_1, \cdots\cdots, X_{a-1}, X_a^*), (X_1, \cdots\cdots X_{a-1}, X_{a+1}, \cdots\cdots X_q)(X_1, \cdots\cdots, X_{a-1})$ be the system of orthogonal unit vectors with the directions of edges of P_a^*, P_b, P_d at z respectively. The unit vector X_a with the direction zx is represented by

$$X_a = u^a X_a^* + u^{a+1} X_{a+1} + \cdots\cdots + u^q X_q \quad \left(\sum_{c=a}^{q} (u^c)^2 = 1\right).$$

By virtue of the definition of $F^{(a)}(X_1, \cdots, X_a)$ if we put $\lambda = |U(A)|^{\mathrm{L}}$, we get

$$\omega^{(a)} = \lambda F^{(a)}(X_1, \cdots\cdots, X_a) = F^{(a)}(X_1, \cdots\cdots, X_{a-1}, \lambda X_a).$$

Hence the $(q-d+1)$-dimensional hypersurface

$$T: \quad F^{(a)}(X_1, \cdots\cdots, X_{a-1}, u^a X_a^* + u^{a+1} X_{a+1} + \cdots\cdots + u^q X_q) = \omega^{(a)}$$

is the locus of the points x such that $zx = |U(A)|^{\mathrm{L}}$, when A varies. If C is strictly convex then so is T, and its order of differentiability with respect to the variables u's is not less than that of C. Therefore the supporting plane of T parallel to B in Q is unique and has only one common point with T, when C is strictly convex. Let this common point be x_0. Then the a-flat A spanned by zx_0 and D is normal to B. The direction perpendicular to this spupporting plane is given by

$$\left(\left(\frac{\partial F^{(a)}}{\partial u^a}\right)_0, \quad \left(\frac{\partial F^{(a)}}{\partial u^{a+1}}\right)_0, \cdots\cdots, \quad \left(\frac{\partial F^{(a)}}{\partial u^q}\right)_0\right)$$

where $(\ \ )_0$ denotes the value at x_0, and this vector is proportional to $(1, 0, \cdots\cdots, 0)$ which coincides with X_a^*. *Hence, in order that* A *is normal to* B, *when* C *is strictly*

convex and differentiable, the condition

$$(2.2)\qquad \begin{cases} \dfrac{\partial \boldsymbol{F}^{(a)}}{\partial \xi_a^i}\ \xi_a^{*i} \neq 0, \\ \dfrac{\partial \boldsymbol{F}^{(a)}}{\partial \xi_a^i}\ \xi_c^i = 0 \quad (c=a+1, \cdots\cdots\cdots, q) \end{cases}$$

is necessary and sufficient. If $a=1$ and $b=n-1$, then B is a hyperplane and $d=0$. In this case, T coincides with C: $F(\xi)=1$ and we have the condition (0.1) as a special case of (2.2).

II $b=d+1$. In this case, let A^*, B, D be spanned by the systems of orthogonal unit vectors $(X_1, \cdots\cdots, X_{b-1}\ X_{b+1}^*, \cdots\cdots, X_q^*)$, $(X_1, \cdots\cdots X_{b-1}, X_b)$, $(X_1, \cdots\cdots, X_{b-1})$ respectively, and let z be a vertex of P_d. The ray zz^* perpendicular to A^* at z is contained in B. Accordingly, we know that zz^* has the same direction as X_b. Let G be the set of all lines perpendicular to A^* at every point of P_a^*. We shall consider in Q the a-box P_a cut off by G on the a-flat A through D. We can easily find that $P_a{}^L = |P_a^*{}^L \sec\theta$, where θ is the angle between the ray zx perpendicular to A at z and the ray zz^*, and $Sm(A, B)$ has the maximal value if and only if $U(A)^L \cos\theta$ is maximal, when A varies and P_b is fixed. This fact easily follows from the definition of $Sm(A, B)$. Therefore we know that an a-flat perpendicular to such ray zx at D is normal to B. Next we shall assume that C is differentiable and strictly convex. Let A be spanned by a system of orthogonal unit vectors $(X_1, \cdots X_{b-1}, X_{b+1}\cdots, X_q)$. Then the vector X perpendicular to A is expressible by

$$(2.3)\qquad X=u^b X_b+u^{b+1} X_{b+1}^* +\cdots\cdots\cdots; +u_q X_q^* \quad \left(\sum_{c=b}^{q} (u^c)^2=1\right).$$

The components of the vector X is also expressible in terms of components of the unit vectors $X_1, X_2, \cdots\cdots\cdots, X_{b-1}, X_{b+1}, \cdots\cdots X_q, Y_1 \cdots\cdots\cdots, Y_{n-q}$ where $Y_1 . \cdots\cdots\cdots, Y_{n-q}$ is a system of orthogonal unit vecotors perpendicular to Q in R. Therefore $F^{(a)}(X_1, \cdots\cdots, X_{b-1}, X_{b+1}, \cdots\cdots\cdots, X_q)$ is a functon on the unit vectors expressed by (2.3). Hence we can put

$$(2.4)\qquad F^{(a)}(X_1, \cdots\cdots, X_{b-1}, X_{b+1}, \cdots\cdots, X_q)$$
$$= \overline{F}^{(a)}(u^b X_b + u^{b+1} X_{b+1}^* + \cdots\cdots\cdots + u^q X_q^*).$$

From this, we can easily prove that $F^{(a)}(X)$ is represented by $F^{(a)}(\xi^1, \cdots\cdots, \xi^n)$ where

$$\xi^i=(-1)^{i-1}\begin{vmatrix} \xi_1^1 & \cdots & \xi_1^n \\ \cdots & \cdots & \cdots \\ \xi_{b-1}^1 & \cdots & \xi_{b-1}^n \\ \cdots & \cdots & \cdots \\ \xi_{b+1}^1 & \cdots & \xi_{b+1}^n \\ \cdots & \cdots & \cdots \\ \eta_1^1 & \cdots & \eta_1^n \\ \cdots & \cdots & \cdots \\ \eta_{n-q}^1 & \cdots & \eta_{n-q}^n \end{vmatrix}$$

in other words, $\boldsymbol{F}^{(a)}(\mathbf{X}_1, \cdots\cdots, \mathbf{X}_{b-1}, \mathbf{X}_{b+1}, \cdots\cdots, \mathbf{X}_q)$ is also a function on $\mathbf{X}=(\xi^1, \cdots, \xi^n)$. Furthermore, $\boldsymbol{F}^{(a)}(\mathbf{X})$ is positve homogeneous of the first order in the variables ξ's. Hence by virtue of the definition of the function $\overline{\boldsymbol{F}}^{(a)}(\mathbf{X})$ we have

$$(2.4)' \qquad \overline{\boldsymbol{F}}^{(a)}(u^b\mathbf{X}_b+u^{b+1}\mathbf{X}_{b+1}^*+\cdots\cdots+u^q\mathbf{X}_q^*)=\omega^{(a)},$$

if we use $(u^b, u^{b+1}, \cdots\cdots, u^q)$ in place of $(\lambda u^b, \lambda u^{b+1}, \cdots\cdots, \lambda u^q)$ where λ denotes $|\boldsymbol{U}(\boldsymbol{A})|^L$. We can easily find that the order of differentiabilty of $\boldsymbol{F}^{(a)}(\mathbf{X}_1 \cdots\cdots, \mathbf{X}_{b-1}, \mathbf{X}_{b+1}, \cdots\cdots, \mathbf{X}_q)$ with respect to the variables ξ's is not less than that of $\boldsymbol{F}^{(a)}(\mathbf{X}_1, \cdots\cdots, \mathbf{X}_{b-1}, \mathbf{X}_{b+1}, \cdots\cdots, \mathbf{X}_q)$ with respect to the vector components.

If $|\boldsymbol{U}(\boldsymbol{A})|^L$ is laid off on the ray zx perpendicular to $\boldsymbol{A}$ from z. the equation (2.4)' is the locus of the end points in $\boldsymbol{Q}$, We shall denote this locus by $\boldsymbol{T}$. If $\boldsymbol{C}$ is strictly convex and differentiable then so is $\boldsymbol{T}$. $\boldsymbol{U}(\boldsymbol{A})^L \cos\vartheta$ is maximal when the supporting plane of $\boldsymbol{T}$ is perpendicular to zz^*. At this time, this supporting plane has only one common point x_0 with $\boldsymbol{T}$. The a-flat perpendicular to zx is normal to $\boldsymbol{B}$ at $\boldsymbol{D}$. This fact yields that the vector

$$(2,5) \qquad \left(\left(\frac{\partial\overline{\boldsymbol{F}}^{(a)}}{\partial u^b}\right)_0, \left(\frac{\partial\overline{\boldsymbol{F}}^{(a)}}{\partial u^{b+1}}\right)_0, \cdots\cdots, \left(\frac{\partial\overline{\boldsymbol{F}}^{(a)}}{\partial u^q}\right)_0\right)$$

is proportional to $(1,0,\cdots\cdots0)$ where $(\ \ )_0$ denotes the value at x_0. *Hence, in order that* $\boldsymbol{A}$ *is normal to* $\boldsymbol{B}$ *the following condition is necessary and sufficient.*

$$(2.6) \qquad \begin{cases} \dfrac{\partial\boldsymbol{F}^a}{\partial\xi^i}\,\xi_b^i\neq 0, \\ \dfrac{\partial\boldsymbol{F}^{(a)}}{\partial\xi^i}\,\xi_c^{*i}=0 \quad (c=b+1, \cdots\cdots, q). \end{cases}$$

W. Barthel did not distinguish the two cases **I, II.** and his theory was also ambiguous because he did not make use of the property of the locus T in **I, II.**

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ON A PROOF OF FUNDAMENTAL FORMULA CONCERNING REFRACTION METHOD OF GEOPHYSICAL PROSPECTING AND SOME REMARKS

Ryuzo ADACHI

(Received August 28, 1954)

1. Introdction

Let M_2, M_3, ·····, M_n be planes which lay uder a horizontal plane M_1 and all parallel to a fixed straight line l. And let s_1, s_2, ·····, s_{n-1} be elastic material and filled between M_1 and M_2, M_2 and M_3, ·····, M_{n-1} and M_n respectively, and s_n under M_n. Now as shown in Fig. 1., let A_1B_1, A_2B_2, ·····, A_n B_n be the intersections of M_1, M_2, ·····, M_n with a plane N_0 which is perpendicular to l, and let $A_1P_2P_3$ ·····P_{n-1} P_n Q_n Q_{n-1} ····· $Q_3Q_2B_1$ be the path along which a wave ray is propagated from A_1 to B_1 and its travel time is T_{An}, refracting at P_2, P_3, ·····, P_{n-1}, Q_{n-1}, ·····, Q_3, Q_2 and at P_n the incident angle is equal to the critical angle and the wave rans along P_n Q_n and at Q_n the wave refracts at the same angle, where $A_1A_n \perp A_1B_1$, $B_1B_n \perp A_1B_1$. In this case, let us take h_{A_i}, h_{B_i}, a_i, b_i (i=1, 2, 3, ·····) as shown in Fig. 1., and w_i be the angles between A_1B_1 and A_iB_i and v_i be the velocities of the wave in s_i. Put

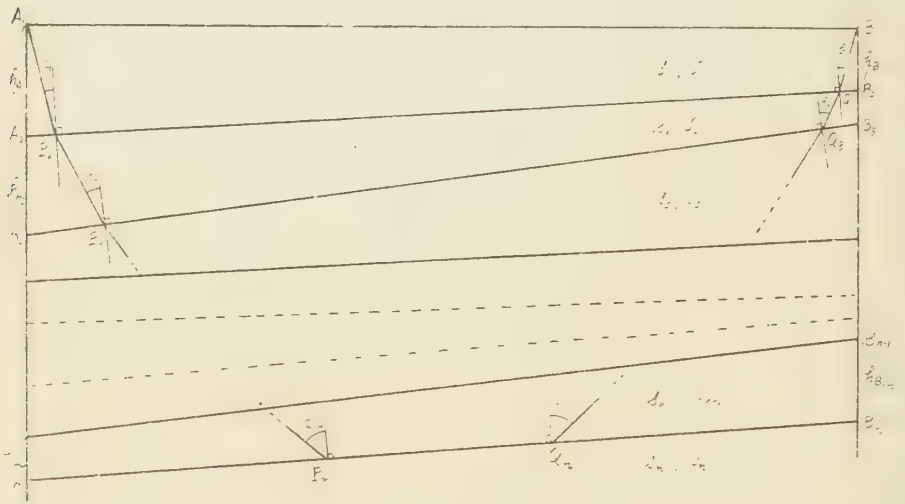

Fig. 1.

$$\overline{A_1 B_1} = x \text{ and } \alpha_i = a_i + w_{i+1},\ \beta_i = b_i - w_{i+1} \quad \cdots\cdots (1)$$

then there exists follwing relation

$$T_{An} = \sum_{i=1}^{n-1} \frac{h_{A_i}}{v_i} (cos\alpha_i + cos\beta_i) + \frac{sin_i\beta_1}{v_1} x \quad \cdots\cdots (2)$$

and this formula is fundamental relation concerning refraction method.

In the following, I will give a proof of this formula.

2. A Proof of the Formula

At first, when $n=2$, (2) becomes

$$T_{A_2} = \frac{h_{A_1}}{v_1} (cos\alpha_1 + cosB_1) + \frac{sin\beta_1}{v_1} x$$

and trueness of this formula is easily shown.

Next suppose (2) is true when n is $n-1$, and as shown in Fig. 1., let a wave ray start from A_1 and reach to B_1 passing through $A_1 P_2 P_3 \cdots\cdots P_{n-1} P_n Q_n Q_{n-1} \cdots\cdots Q_3 Q_2 B_1$. As shown in Fig. 2., let L be the intersection of $A_1 A_n$ and the elongation of $P_3 P_2$ and let N, M be the intersection of the straight line passing L and parallel to $A_1 B_1$ with $B_1 B_n$, the elongation of $Q_3 Q_2$ respectively. Let us suppose that the space between LM and $A_2 B_2$ is filled with s_2 in place of s_1, then it is evident that a wave ray passes from L to M passing through $LP_2 P_3 \cdots\cdots P_{n-1} P_n Q_n Q_{n-1} \cdots\cdots Q_3 Q_2 M$ and let T'_{A_n} be the time which is spent, then as (2) is satisfied when n is $n-1$, we have

$$T'_{A_n} = \frac{h_{A_2} + LA_2}{v_2} (cos\alpha_2 + cos\beta_2) + \sum_{i=3}^{n-1} \frac{h_{A_i}}{v_i} (cos\alpha_i + cos\beta_i)$$

$$+ \frac{sin_i\beta_2}{v_2} x' \quad \cdots\cdots (3)$$

where $x' = LM = x - MN$.

while

$$T_{An} = T'_{An} + \frac{1}{v_1} (A_1 P_2 + B_1 Q_2) - \frac{1}{v_2} (LP_2 + MQ_2),$$

therefore if we put

$$T' = \sum_{i=2}^{n-1} \frac{h_{A_i}}{v_i} (cos\alpha_i + cos\beta_i) \quad \cdots\cdots (4)$$

$$\Delta = A_1P_2+B_1Q_2+\frac{v_1}{v_2}\left\{LA_2(cos\alpha_2+cos\beta_2)+x'\,sin\beta_2-LP_2-MQ_2\right\}\cdots\cdots (5)$$

we get

$$T_{An} = T' + \frac{1}{v_1}\cdot\Delta \cdots\cdots (6).$$

Now it is easily shown that

$$MN=Q_2G-Q_2S=\frac{cosw_2\,sin\beta_1}{cosb_1}h_{A_1}-\frac{sinw_2\,sin\beta_1}{cosb_1}x-MS\,tan\beta_2$$

and

$$x'=\left(\frac{sinw_2\,sin\beta_1}{cosb_1}+1\right)x-\frac{cosw_2\,sin\beta_1}{cosb_1}h_{A_1}+MS\,tan\beta_2$$

therefore

$$\Delta=\frac{cosw_2}{cosa_1}h_{A_1}+\frac{cosw_2}{cosb_1}h_{A_1}-\frac{sinw_2}{cosb_1}x$$

$$+\frac{v_1}{v_2}\left[\frac{cos(\alpha_2-w_2)\,sin\alpha_1}{sin\alpha_2\,cosa_1}(cos\alpha_2+cos\beta_2)h_{A_1}+sin\beta_2\left\{\left(\frac{sinw_2\,sin\beta_1}{cosb_1}+1\right)x\right.\right.$$

$$\left.\left.-\frac{cosw_2\,sin\beta_1}{cosb_1}h_{A_1}+MStan\beta_1\right\}-\frac{cosw_2\,sin\alpha_1}{sin\alpha_2\,cosa_1}h_{A_1}-\frac{MS}{cos\beta_2}\right],$$

and arrangement of right side of this equation applying the relation

$$\frac{v_1}{v_2}=\frac{sina_1}{sin(\alpha_2-w_2)}=\frac{sinb_1}{sin(\beta_2+w_2)}$$

leads the following result

$$\Delta = h_{A_1}(cos\alpha_1+cos\beta_1)+xsin\beta_1.$$

Consequently from (6) we have

$$T_{An}=\sum_{i=1}^{n-1}\frac{h_{A_i}}{v_i}(cos\alpha_i+cos\beta_i)+\frac{sin\beta_1}{v_1}x$$

this indicates that the relation (2) is satisfied when n is n, therefore by mathematical induction, (2) is generally true.

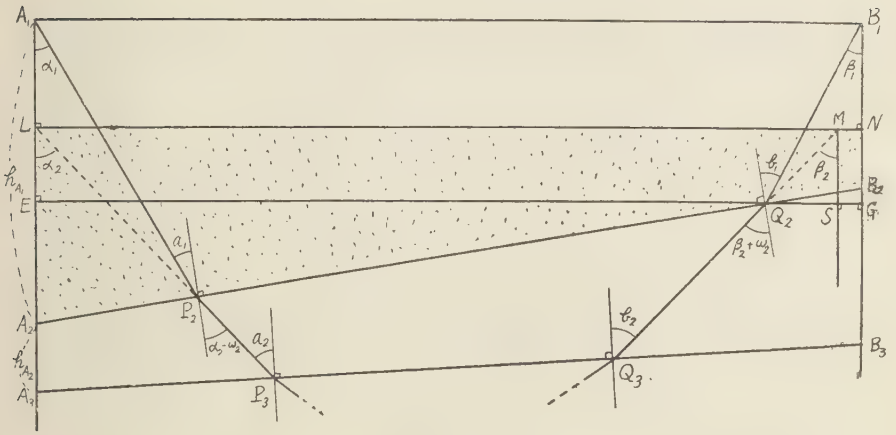

Fig. 2.

3. Application of the Formula

If $w_2, w_3, \cdots\cdots, w_{n-1}$; $v_1, v_2, \cdots\cdots, v_{n-1}$; $h_{A_1}, h_{A_2}, \cdots\cdots, h_{A_{n-2}}$; $h_{B_1}, h_{B_2}, \cdots\cdots, h_{B_{n-2}}$ and travel time T_{A_n}, T_{B_n} and v_{A_n}, v_{B_n} which are the velocities in the traval time curve, are known and from these values, we consider how to find v_n, w_n, $h_{A_{n-1}}$, $h_{B_{n-1}}$.

From the formula (2) we get

$$T_{A_n} = \sum_{i=1}^{n-1} \frac{h_{A_i}}{v_i} (cos\alpha_i + cos\beta_i) + \frac{sin\beta_1}{v_1} x \cdots\cdots \text{(i)}$$

$$T_{B_n} = \sum_{i=1}^{n-1} \frac{h_{B_i}}{v_i} (cos\alpha_i + cos\beta_i) + \frac{sin\alpha_1}{v_1} x \cdots\cdots \text{(ii)}$$

and the coefficients of x are $v_{A_n}^{-1}$, $v_{B_n}^{-1}$, therefore

$$sin\beta_1 = \frac{v_1}{v_{A_n}}, \quad sin\alpha_1 = \frac{v_1}{v_{B_n}} \cdots\cdots (7).$$

Next by the law of refraction, we get

$$\frac{v_1}{v_2} = \frac{sina_1}{sin(\alpha_2 - w_2)} = \frac{sinb_1}{sin(\beta_2 + w_2)}$$

$$\frac{v_2}{v_3} = \frac{sina_2}{sin(\alpha_3 - w_3)} = \frac{sinb_2}{sin(\beta_3 + w_3)}$$

.....

.....

$$\frac{v_{n-2}}{v_{n-1}} = \frac{sina_{n-2}}{sin(\alpha_{n-1} - w_{n-1})} = \frac{sinb_{n-2}}{sin(\beta_{n-1} + w_{n-1})}$$

$$\frac{v_{n-1}}{v_n} = sina_{n-1} = sinb_{n-1}$$

therefore

$$sin\,(b_1 - w_2) = \frac{v_1}{v_{A_n}}, \quad sin\,(a_1 + w_2) = \frac{v_1}{v_{B_n}}$$

$$\frac{sina_1}{sin\,(a_2 + w_3 - w_2)} = \frac{sinb_1}{sin\,(b_2 - w_3 + w_2)} = \frac{v_1}{v_2}$$

$$\frac{sina_2}{sin\,(a_3 + w_4 - w_3)} = \frac{sinb_2}{sin\,(b_3 - w_4 + w_3)} = \frac{v_2}{v_3} \qquad \cdots\cdots\cdots\cdots (8)$$

.....

.....

$$\frac{sina_{n-2}}{sin(a_{n-1} + w_n - w_{n-1})} = \frac{sinb_{n-2}}{sin\,(b_{n-1} - w_n + w_{n-1})} = \frac{v_{n-2}}{v_{n-1}}$$

$$a_{n-1} = b_{n-1}, \qquad \frac{v_{n-1}}{v_n} = sin\ a_{n-1}$$

from these relations a_1, b_1; a_2, b_2; ……; a_{n-2}, b_{n-2}; a_{n-1}, w_n, v_n can be calculated successively and then from (2), $h_{A_{n-1}}$, $h_{B_{n-1}}$ are obtained.

4. Some remarks

When we calculate the values of v_n, w_n, $h_{A_{n-1}}$ etc., *in some books, it is written thaht*

" If $\Theta_{k-1,k}$ is the critical angle at A_k B_k line, then there are the relations

$$sin\,(\Theta_{k-1,k} - w_k) = \frac{v_{k-1}}{v_{A_k}}, \quad sin\,(\Theta_{k-1,k} + w_k) = \frac{v_{k-1}}{v_{B_k}},$$

and by using these relations, calculating formulas of v_n, w_n etc., are derived "

But these relations do not be satisfied in general, for example if we put $k = 3$ and suppose

$$sin\,(\Theta_{2.3} - w_3) = \frac{v_2}{v_{A_3}} \quad \cdots\cdots\cdots\

ON THE NUMERICAL SOLUTION OF THE SIMULTANEOUS DIFFERENTIAL EQUATIONS UNDER SOME CONDITIONS (Part. 2)

Ryuzo ADACHI

(Received August 28, 1954)

1. Introduction

In the previous Journal (Vol. 1, No. 4), I have treated of a general method of the numerical solution of the simultaneous differential equations

$$\frac{du_j}{dx} = F_j(x, u_1, u_2, u_3) \quad \cdots\cdots (1)$$
$$j = 1, 2, 3$$

under some conditions.

In this place, as these conditions, I will take the following cases which appear in practical problems frequently.

(i) $u_1 = 0$ when $x = 0$, $u_1 = A_2$ when $x = a_2$, $u_2 = A_3$ when $x = a_3$

(ii) $u_1 = 0$ when $x = 0$, $u_2 = K(u_1)$ when $x = a_2$, $u_3 = H(u_1, u_2)$ when $x = a_3$

(iii) $u_1 = 0$ when $x = 0$,

$u_3 = H_2(x, u_1)$ ····· at the intersection of the solution curve $u = u_2(x)$ and a given curve $u = \Psi_2(x)$

$u_2 = H_3(x, u_1)$ ····· at the intersection of the solution curve $u = u_3(x)$ and a given curve $u = \Psi_3(x)$

where a_2, a_3, A_2, A_3 are given constants and $K(u_1)$, $H(u_1, u_2)$, $H_2(x, u_1)$, $H_3(x, u_1)$ are given functions. In the following, the same symbols which were employed in the previous Journal are used.

2. Case (i)

At first, putting the conditions $u_1(0) = 0$, $u_2(a_3) = A_3$ in the relations

$$u_{1,n}(x) = p_{1,n}(x) + b_{1,n}, \quad u_{2,n}(x) = p_{2,n}(x) + b_{2,n}$$

we have

$$b_{1,n} = 0, \quad b_{2,n} = A_3 - P_{2,n}(a_3).$$

Next, from the condition $u_1(a_2) = A_2$, we have

$$A_2 = u_{1,n}(a_2) = u_{1,n+1}(a_2) = \int_0^{a_2} F_1\big(x,\ p_{1,n}(x),\ p_{2,n}(x)$$

$$+ b_{2,n},\ p_{3,n}(x) + b_{3,n}\big)\,dx$$

that is

$$\int_0^{a_2} F_1\big(x,\ p_{1,n}(x),\ p_{2,n}(x) + A_3 - p_{2,n}(a_3),\ p_{3,n}(x) + b_{3,n}\big)\,dx = A_2\,,$$

and solving this relation with respect to $b_{3,n}$ numerically, we can determine its value.

3. Case (ii)

Similarly we have

$$b_{1,n} = 0\,, \quad p_{2,n}(a_2) + b_{2,n} = K\big(p_{1,n}(a_2)\big)$$

$$p_{3,n}(a_3) + b_{3,n} = H\big(p_{1,n}(a_3),\ p_{2,n}(a_3) + b_{2,n}\big)$$

and solviog these relations with respect to $b_{2,n}$, $b_{3,n}$ we can find their values.

4. Caes (iii)

In this case we have $b_{1,n} = 0$ and

$$p_{2,n}({}_nX_2) + b_{2,n} = \Psi_2({}_nX_2), \qquad p_{3,n}({}_nX_2) + b_{3,n} = H_2\,[{}_nX_2\,,\ p_{1,n}({}_nX_2)]$$

$$p_{3,n}({}_nX_3) + b_{3,n} = \Psi_3({}_nX_3), \qquad p_{2,n}({}_nX_3) + b_{2,n} = H_3\,[{}_nX_3\,,\ p_{1,n}({}_nX_3)].$$

Therefore putting

$$A_n(x) = \Psi_2(x) - p_{2,n}(x), \qquad B_n(x) = H_3\,[x,\ p_{1,n}(x)] - p_{2,n}(x)$$

$$D_n(x) = \Psi_3(x) - p_{3,n}(x), \qquad E_n(x) = H_2\,[x,\ p_{1,n}(x)] - p_{3,n}(x)$$

gives

$$b_{2,n} = A_n({}_nX_2) = B_n({}_nX_3), \qquad b_{3,n} = D_n({}_nX_3) = E_n({}_nX_2),$$

and if we put ${}_nX_3 = B_n^{-1}\,[A_n({}_nX_2)] = G_n({}_nX_2)$ we have $D_n\,[G\ ({}_nX_2)] = E_n({}_nX_2)$,

therefore ${}_nX_2$ can be determined as a root of the equation

$$S_n(x) \equiv D_n\,[G_n(x)] - E_n(x) = 0$$

and then $b_{2,n}$, $b_{3,n}$ are determined by $b_{2,n} = A_n({}_nX_2)$, $b_{3,n} = E_n({}_nX_2)$.

5. Examples

(1) Solve the equations

$$\frac{du_1}{dx} = \frac{1}{1+u_1^2}, \quad \frac{du_2}{du} = \frac{1}{1+u_1^2+u_2^2}, \quad \frac{du_3}{du} = \frac{1}{1+u_2^2+u_3^2}$$

under the conditions

$$u_1(0) = 0, \quad u_1(1) = u_2(1), \quad u_3(1,5) = u_1(1,5) + u_2(1,5).$$

This problem is in the case (ii), and

$$a_2 = 1, \quad K(u_1) = u_1, \quad a_3 = 1,5, \quad H(u_1, u_2) = u_1 + u_2.$$

Therefore

$$f_{1,n}(x) = \frac{1}{1+u_{1,n}^2(x)}, \; f_{2,n}(x) = \frac{1}{1+u_{1,n}^2(x)+u_{2,n}^2(x)},$$

$$f_{3,n} = \frac{1}{1+u_{2,n}^2(x)+u_{3,n}^2(x)}$$

$$p_{1,n}(x) = \int_0^x f_{1,n-1}(x)\,dx, \quad p_{2,n}(x) = \int_0^x f_{2,n-1}(x)\,dx,$$

$$p_{3,n}(x) = \int_0^x f_{3,n-1}(x)\,dx,$$

and

$$u_{1,n}(x) = p_{1,n}(x) + b_{1,n}, \quad u_{2,n}(x) = p_{2,n}(x) + b_{2,n},$$
$$u_{3,n}(x) = p_{3,n}(x) + b_{3,n}$$

and from the conditions, we have

$$b_{1,n} = 0, \quad b_{2,n} = u_{1,n}(1) - p_{2,n}(1),$$
$$b_{3,n} = u_{1,n}(1,5) + u_{2,n}(1,5) - p_{3,n}(1,5).$$

Starting from $u_{1,1}$, $u_{2,1}$, $u_{3,1}$ which were arbitrarily given, the result of our calculation is shown in the tables. 1, 2, 3 and as they indicate, $u_{1,8} \equiv u_{1,9}$, $u_{2,8} \equiv u_{2,9}$, $u_{3,8} \equiv u_{3,9}$ are satisfied except few points which are marked by *.

Table. 1.

x	$f_{1.1}$	$p_{1.2}=u_{1.2}$	$f_{2.1}$	$p_{2.2}$	$u_{2.2}$	$f_{3.1}$	$p_{3.2}$	$u_{3.2}$	$f_{1.2}$	$p_{1.3}=u_{1.3}$	$f_{2.2}$	$p_{2.3}$
0.0	0.500	0.000	0.400	0.000	0.100	0.500	0.000	0.700	1.000	0.000	1.000	0.000
0.1	500	0.050	400	0.040	140	500	0.050	750	0.998	0.100	0.979	0.099
0.2	500	0.100	400	0.080	180	500	0.100	800	990	199	960	0.196
0.3	500	150	400	0.120	220	500	150	850	978	298	934	291
0.4	500	200	400	160	260	500	200	900	962	395	903	382
0.5	500	250	400	200	300	500	250	0.950	941	490	867	471
0.6	500	300	400	240	340	500	300	1.000	917	583	829	558
0.7	500	350	400	280	380	500	350	1.050	891	673	789	637
0.8	500	400	400	320	420	500	400	1.100	862	761	749	714
0.9	500	450	400	360	460	500	450	1.150	831	845	707	785
1.0	500	500	400	400	500	500	500	1.200	800	0.927	667	855
1.1	500	550	400	440	540	500	550	1.250	769	1.005	627	920
1.2	500	600	400	480	580	500	600	1.300	735	1.081	580	0.981
1.3	500	650	400	520	620	500	650	1.350	703	1.152	553	1.038
1.4	500	700	400	560	660	500	700	1.400	671	1.221	519	1.091
1.5	500	750	400	600	700	500	750	1.450	640	1.287	487	1.142
1.6	500	800	400	640	740	500	800	1.500	610	1.349	457	1.189
1.7	500	850	400	680	780	500	850	1.550	580	1.409	429	1.233
1.8	500	900	400	720	820	500	900	1.600	553	1.465	403	1.275
1.9	500	0.950	400	760	860	500	0.950	1.650	526	1.519	379	1.314
2.0	0.500	1.000	0.400	0.800	0.900	0.500	1.000	1.700	0.500	1.571	0.356	1.351

$u_{1.1}=1$, $u_{2.1}=u_{3.1}=\dfrac{1}{\sqrt{2}}$

$b_{2.2}=u_{1.2}(1)-p_{2.2}(1)=0.100$

$b_{3.2}=u_{1.2}(1.5)+u_{2.2}(1.5)-p_{3.2}(1.5)=0.700$

Table. 2.

x	$u_{2.3}$	$f_{3.2}$	$p_{3.3}$	$u_{3.3}$	.....	.....	.....	$f_{1.7}$	$p_{1.8}=u_{1.8}$	$f_{2.7}$	$p_{2.8}$
0.0	0.072	0.667	0.000	1.832	.....	.....	.....	1.000	0.000	0.981	0.000
0.1	0.171	632	0.065	1.836				0.990	0.100	940	0.096
0.2	268	598	0.127	1.958				962	197	874	187
0.3	362	565	185	2.016				922	291	799	270
0.4	454	533	240	2.071				873	381	726	347
0.5	543	502	291	2.122	.....	.....	.....	822	466	657	416
0.6	628	473	339	2.171	.....	.....	.....	770	546	596	479
0.7	709	445	385	2.217				723	620	544	536
0.8	785	419	429	2.260				678	690	500	588
0.9	858	395	469	2.301				636	756	461	636
1.0	927	372	508	2.339	.....	.....	.....	600	818	428	680
1.1	0.992	350	544	2.375	.....	.....	.....	566	876	299	722
1.2	1.052	331	578	2.409				536	931	374	760
1.3	1.110	312	610	2.441				508	0.983	352	797
1.4	1.163	295	640	2.472				484	1.033	333	831
1.5	1.214	278	669	2.500	.....	.....	.....	461	1.080	315	863
1.6	1.261	263	696	2.527	.....	.....	.....	441	1.125	300	894
1.7	1.305	249	722	2.553				423	1.169	286	923
1.8	1.347	236	746	2.577				405	1.210	274	951
1.9	1.386	224	769	2.600				390	1.250	262	0.978
2.0	1.422	0.213	0.791	2.622	.....	.....	.....	0.376	1.288	0.252	1.004

$b_{2.3}=u_{1.3}(1)-p_{2.3}(1)=0.072$

$b_{3.3}=u_{1.3}(1.5)+u_{2.3}(1.5)-p_{3.3}(1.5)=1.832$

Table. 3.

x	$u_{2.8}$	$f_{3.7}$	$p_{3.8}$	$u_{3.8}$	$f_{1.8}$	$p_{1.9}=u_{1.9}$	$f_{2.8}$	$p_{2.9}$	$u_{2.9}$	$f_{3.8}$	$p_{3.9}$	$u_{3.9}$
0.0	0.138	0.236	0.000	1.793	1.000	**0.000**	0.981	0.000	**0.138**	0.236	0.000	**1.793**
0.1	234	230	0.023	1.816	0.990	**0.100**	939	0.096	**234**	230	0.023	**1.816**
0.2	324	223	0.046	1.839	962	**197**	874	0.187	**325***	223	0.046	**1.839**
0.3	408	216	0.068	1.861	922	**291**	799	270	**408**	216	0.068	**1.861**
0.4	484	209	0.089	1.882	873	**381**	725	347	**485***	209	0.089	**1.882**
0.5	554	203	0.110	1.903	822	**466**	657	416	**554**	203	0.100	**1.903**
0.6	616	197	130	1.923	770	**546**	596	478	**616**	197	130	**1.923**
0.7	673	191	150	1.942	723	**620**	544	535	**673**	191	149	**1.942**
0.8	725	186	168	1.961	678	**690**	500	588	**726***	186	168	**1.961**
0.9	773	181	186	1.979	636	**756**	461	636	**774***	181	186	**1.979**
1.0	818	177	204	1.997	600	**818**	428	680	**818**	177	204	**1.997**
1.1	859	172	222	2.015	566	**876**	399	721	**859**	172	222	**2.015**
1.2	898	168	239	2.032	536	**931**	374	760	**898**	168	239	**2.032**
1.3	934	165	255	2.048	509	**0.984***	352	796	**934**	165	255	**2.048**
1.4	0.968	161	272	2.065	484	**1.033**	333	831	**0.969***	161	272	**2.065**
1.5	1.001	158	288	2.081	462	**1.080**	316	863	**1.001**	158	288	**2.081**
1.6	1.032	155	303	2.096	441	**1.125**	300	894	**1.032**	155	303	**2.096**
1.7	1.061	152	319	2.112	422	**1.168***	286	923	**1.061**	152	319	**2.112**
1.8	1.089	149	334	2.127	406	**1.210**	274	951	**1.089**	149	334	**2.127**
1.9	1.116	146	348	2.141	390	**1.250**	263	978	**1.116**	146	348	**2.141**
2.0	1.141	0.144	0.363	2.156	0.376	**1.288**	0.252	1.004	**1.142***	0.144	0.363	**2.156**

$b_{2.8}=u_{1.8}(1)-p_{2.8}(1)=0.138$ $\quad b_{2.9}=u_{1.9}(1)-p_{2.9}(1)=0.138$

$b_{3.8}=u_{1.8}(1.5)+u_{2.8}(1.5)-p_{3.8}(1.5)=1.793$ $\quad b_{3.9}=u_{1.9}(1.5)+u_{2.9}(1.5)-p_{3.9}(1.5)=1.793$

(2) Solve the equations

$$\frac{du_1}{dx}=\frac{1}{5+u_2^2},\quad \frac{du_2}{dx}=\frac{1}{6+u_3^2},\quad \frac{du_3}{dx}=\frac{1}{4+u_1^2}$$

under the conditions

$$u_1(0)=0$$

$u_3(x)=x+0.525$ ····· at the intersection of the solution curve $u=u_2(x)$ and the straight line $u=0.500$

$u_2(x)=2x-2.725$ ····· at the intersection of the solution curve $u=u_3(x)$ and the straight line $u=x$.

This problem is in the case (iii), and

$$\Psi_2(x)=0.500,\quad H_2(u,u_1)=x+0.525$$

$$\Psi_3(x)=x,\quad H_3(x,u_1)=2x-2.725$$

and

$$A_n(x)=0.500-p_{2,n}(x),\quad B_n(x)=2x-2.725-p_{2,n}(x)$$

$$D_n(x)=x-p_{3,n}(x),\quad E_n(x)=x+0.525-p_{3,n}(x)$$

$$b_{1,n}=0,\quad b_{2,n}=A_n({}_nX_2),\quad b_{3,n}=E_n({}_nX_2)$$

where ${}_nX_2$ is a root of

$$S_n(x) \equiv D_n[G_n(x)] - E_n(x) = 0.$$

Starting from $u_{1,1}$, $u_{2,1}$, $u_{3,1}$ which were arbitrarily given, the result of our calculation is shown in the tables. 4, 5 and they indicate good convergence of the solutions.

Tacle. 4.

x	$u_{1.3}$	$u_{2.3}$	$u_{3.3}$	$f_{1.3}$	$u_{1.4}$	$f_{2.3}$	$p_{2.4}$	$f_{3.3}$	$p_{3.4}$	A_4
0.0	0.000	0.378	1.242	0.195	0.000	0.133	0.000	0.250	0.000	
0.1	0.019	391	1.267	194	0.020	131	0.013	250	0.025	
0.2	39	404	1.292	194	39	130	26	250	50	
0.3	58	418	1.317	193	58	129	39	250	75	
0.4	78	430	1.342	193	78	128	52	250	0.100	
0.5	0.097	443	1.367	192	0.097	127	65	250	125	
0.6	0.116	456	1.392	192	0.116	126	78	250	150	
0.7	135	468	1.417	192	135	125	90	249	175	
0.8	154	481	1.442	191	154	124	0.103	249	200	
0.9	173	493	1.467	191	173	123	115	248	225	0.385
1.0	192	505	1.492	190	193	122	127	248	250	0.373
1.1	211	518	1.516	190	212	121	139	248	274	0.361
1.2	230	530	1.541	189	230	120	151	247	299	
1.3	249	542	1.566	189	249	118	163	247	324	
1.4	268	553	1.590	188	268	117	175	246	348	
1.5	287	565	1.615	188	287	116	187	245	373	
1.6	306	577	1.639	188	306	115	198	244	397	
1.7	324	588	1.664	187	325	114	210	243	422	
1.8	343	599	1.688	187	343	113	221	243	446	
1.9	362	611	1.712	186	362	112	232	242	470	
2.0	0.380	0.622	1.737	0.186	0.381	0.111	0.243	0.242	0.495	

$u_{1.1} = 0$, $u_{2.1} = 1 - \frac{x}{2}$, $u_{3.1} = 1.5$ $\quad {}_2X_2 = 0.951$, $b_{2.2} = A_2(0.951) = 0.385$,

$b_{3.2} = E_2(0.951) = 1.238$

Table. 5.

x	B_4	G_4	D_4	$D_4(G_4)$	E_4	S_4	$u_{2.4}$	$u_{3.4}$	.....	$u_{1.5}$	$u_{2.5}$	$u_{3.5}$
0.0							0.378	1.242		0.000	0.378	1.242
0.1							391	1.267		0.020	391	1.267
0.2							404	1.292		39	404	1.292
0.3							417	1.317		58	417	1.317
0.4							430	1.342		78	430	1.342
0.5							443	1.367		0.097	443	1.367
0.6							456	1.392		0.116	456	1.392
0.7							468	1.417		135	468	1.417
0.8							481	1.442		154	481	1.442
0.9		1.658		1.244	1.200	0.044	493	1.467		173	493	1.467
1.0		1.652		1.240	1.275	−0.035	505	1.492		193	505	1.492
1.1		1.645		1.235	1.351	−0.116	517	1.516		212	517	1.516
1.2							529	1.541		230	529	1.541
1.3							541	1.566		249	541	1.566
1.4							553	1.590		268	553	1.590
1.5	0.088		1.127				565	1.615		287	565	1.615
1.6	0.277		1.203				576	1.639		306	576	1.639
1.7	0.465		1.278				588	1.664		325	588	1.664
1.8							599	1.688		343	599	1.688
1.9							610	1.712		362	610	1.712
2.0							0.621	1.737		0.381	0.621	1.737

${}_4X_2 = 0.956$, $b_{2.4} = A_4(0.956) = 0.378$, $b_{3.4} = E_4(0.956) = 1.242$,

${}_5X_2 = 0.956$, $b_{2.5} = A_5(0.956) = 0.378$, $b_{3.5} = E_5(0.956) = 1.242$

(3) Solve the equations

$$\frac{du_1}{dx} = u_1 + \frac{1}{2}u_3, \quad \frac{du_2}{dx} = u_1, \quad \frac{du_3}{dx} = \frac{-2}{u_1 + u_2}$$

under the conditions

$u_1(0) = 0$

$u_3(x) = x$ ..... at the intersection of the solution curve $u = u_2(x)$ and the straight line $u = 1.380$

$u_2(u) = u_1 + 0.240$ ..... at the intersection of the solution curve $u = u_3(x)$ and the straight line $u = x - 0.900$.

This problem is in the case (iii) also, and

$$\Psi_2(x) = 1.380, \qquad H_2(x, u_1) = x$$
$$\Psi_3(x) = x - 0.900, \qquad H_3(x, u_1) = u_1 + 0.240$$

therefrore

$$A_n(x) = 1.380 - p_{2,n}(x), \qquad B_n(x) = p_{1,n}(x) - p_{2,n}(x) + 0.240$$
$$D_n(x) = x - p_{3,n}(x) - 0.900, \quad E_n(x) = x - p_{3,n}(x)$$
$$b_{1,n} = 0, \quad b_{2,n} = A_n({}_nX_2), \quad b_{3,n} = E_n({}_nX_2)$$

where ${}_nX_2$ is a root of

$$S_n(x) = D_n[G_n(x)] - E_n(x) = 0 \quad .$$

Starting from $u_{1,1}$, $u_{2,1}$, $u_{3,1}$ the result of our calculation is shown in the tables. 6, 7, 8, 9, 10 and as they indicate, the convergence of the solutions are not rapid.

Table. 6.

x	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	.....	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	.....	$f_{1,2}$	$p_{1,3}=u_{1,3}=f_{2,3}$	.....
0.0	0.00	1.10	1.80	.....	0.000	1.03	1.65	.....	0.825	0.000	.....
0.1	0.13	1.12	1.72		0.095	1.04	1.48		835	0.083	
0.2	26	1.14	1.63		0.198	1.06	1.33		863	0.168	
0.3	39	1.16	1.54		310	1.09	1.19		905	256	
0.4	53	1.19	1.45		431	1.14	1.07		0.966	350	
0.5	66	1.24	1.37	.....	561	1.19	0.96	.....	1.041	450	.....
0.6	79	1.28	1.28	.....	700	1.27	86	.....	1.130	559	.....
0.7	0.93	1.34	1.20		0.848	1.35	76		1.228	677	
0.8	1.06	1.41	1.11		1.005	1.45	68		1.345	805	
0.9	1.19	1.50	1.02		1.171	1.57	60		1.471	0.943	
1.0	1.32	1.60	0.94		1.346	1.63	53		1.611	1.100	
1.1	1.46	1.71	85	.....	1.529	1.83	46	.....	1.759	1.269	.....
1.2	1.59	1.83	76	.....	1.722	1.98	40	.....	1.922	1.453	.....
1.3	1.72	1.95	67		1.923	2.15	35		2.098	1.654	
1.4	1.85	2.07	59		2.134	2.33	30		2.284	1.873	
1.5	2.00	2.19	50		2.354	2.52	25		2.479	2.111	
1.6	2.13	2.32	41		2.583	2.73	20		2.683	2.369	
1.7	2.25	2.45	0.33	.....	2.820	2.94	0.15	.....	3.895	2.648	.....

${}_1X_2 = 0.724$, $b_{2,2} = A_2(0.724) = 1.[illegible]$, $b_{3,2} = E_2(0.724) = 1.65$

Table. 7.

x	$u_{2.3}$	$u_{3.3}$	$f_{1.3}$	$p_{1.4}=u_{1.4}=f_{2.4}$	$p_{2.4}$	$f_{3.3}$	$p_{3.4}$	A_3	B_3	G_3	E_3	D_3
0.0	0.84	2.12	1.060	0.000	0.000	−2.38	0.000					
0.1	85	1.93	1.048	0.105	0.004	−2.15	−0.227					
0.2	86	1.77	1.053	211	0.017	−1.94	−0.431					
0.3	88	1.62	1.066	316	0.038	−1.75	−0.616					
0.4	92	1.48	1.090	424	0.068	−1.58	−0.782					
0.5	0.97	1.36	1.130	535	0.108	−1.41	−0.932					
0.6	1.03	1.25	1.184	651	159	−1.26	−1.065					
0.7	1.11	1.15	1.237	772	221	−1.12	−1.184	1.159		1.557	1.884	0.984
0.8	1.20	1.07	1.340	0.901	295	−0.99	−1.290	1.085	0.846	1.338	2.090	1.190
0.9	1.31	0.99	1.441	1.040	382	−0.89	−1.384	0.998	0.898	1.178	2.284	1.384
1.0	1.43	92	1.560	1.190	484	−0.79	−1.468	0.896	0.946	0.896	2.468	1.568
1.1	1.58	86	1.699	1.353	603	−0.70	−1.542		0.990		2.642	1.742
1.2	1.74	80	1.853	1.531	739	−0.63	−1.609		1.032		2.809	1.909
1.3	1.92	75	2.029	1.725	0.894	−0.56	−1.668		1.071		2.968	2.068
1.4	2.13	70	2.223	1.937	1.071	−0.50	−1.721		1.106		3.121	2.221
1.5	2.35	66	2.441	2.170	1.270	−0.45	−1.769		1.140		3.269	2.369
1.6	2.60	62	2.679	2.426	1.494	−0.40	−1.811		1.172		3.411	2.511
1.7	2.87	0.59	2.943	2.708	1.745	−0.36	−1.849		1.203		3.549	2.645

$${}_3X_2 = 0.956, \quad b_{2.3} = A_3\,(0.956) = 0.842, \quad b_{3.3} = E_3\,(0.956) = 2.12$$

Table. 8.

x	$D_4(G_4)$	S_4	$u_{2.4}$	$u_{3.4}$	.....	$u_{1.8}$	$u_{2.8}$	$u_{3.8}$	$f_{1.8}$	$p_{1.9}=u_{1.9}=f_{2.9}$	$p_{2.9}$	$f_{3.8}$
0.0			1.079	2.104		0.000	0.989	2.012	1.006	0.000	0.000	−2.022
0.1			1.079	1.877		0.100	0.994	1.820	1.010	0.101	0.005	−1.828
0.2			1.083	1.673		201	1.009	1.647	1.025	203	0.020	−1.653
0.3			1.096	1.488		305	1.034	1.490	1.050	306	0.045	−1.494
0.4			1.147	1.322		411	1.070	1.349	1.085	413	0.081	−1.354
0.5			1.187	1.172		521	1.111	1.221	1.132	524	0.128	−1.225
0.6			1.238	1.039		637	1.175	1.105	1.189	640	186	−1.104
0.7	2.451	0.567	1.300	0.920		758	1.245	1.000	1.258	762	255	−0.999
0.8	2.130	0.040	1.374	814		0.888	1.327	0.906	1.341	0.892	338	−0.903
0.9	1.774	−0.510	1.461	720		1.027	1.424	820	1.437	1.031	434	−0.816
1.0	1.378	−1.090	1.563	636		1.176	1.535	743	1.548	1.180	544	−0.738
1.1			1.682	562		1.337	1.661	673	1.673	1.342	669	−0.667
1.2			1.818	495		1.512	1.804	610	1.817	1.516	812	−0.603
1.3			1.973	436		1.701	1.966	553	1.978	1.706	0.972	−0.545
1.4			2.150	383		1.908	2.148	501	2.158	1.913	1.153	−0.493
1.5			2.349	335		2.135	2.351	454	2.362	2.139	1.355	−0.446
1.6			2.573	293		2.382	2.577	412	2.588	2.386	1.581	−0.403
1.7			2.824	0.255		2.654	2.830	0.374	2.841	2.658	1.833	−0.365

$${}_4X_2 = 0.807, \quad b_{2.4} = A_4\,(0.807) = 1.079, \quad b_{3.4} = E_4\,(0.807) = 2.104$$

Table. 9.

x	$p_{3.9}$	A_9	B_9	G_9	E_9	D_9	$D_9(G_9)$	S_9	$u_{2.9}$	$u_{3.9}$	$f_{1.9}$	$p_{1.10}=u_{1.10}=f_{2.10}$
0.0	0.000								0.997	2.004	1.002	**0.000**
0.1	−0.193								1.002	1.811	1.007	**0.101**
0.2	−0.387								1.017	1.637	1.021	**202**
0.3	−0.524								1.042	1.480	1.046	**305**
0.4	−0.666								1.078	1.338	1.082	**412**
0.5	−0.795								1.125	1.209	1.129	**522**
0.6	−0.912								1.183	1.092	1.186	**638**
0.7	−1.017								1.252	0.987	1.255	**760**
0.8	−1.112	1.042	0.794	1.586	1.912	1.012	2.285	0.374	1.335	892	1.338	**0.890**
0.9	−1.198	0.946	0.837	1.207	2.098	1.198	1.718	−0.379	1.431	806	1.434	**1.028**
1.0	−1.276	0.836	0.876	0.898	2.276	1.376	1.195	−1.080	1.541	728	1.544	**1.177**
1.1	−1.346		0.913			1.546			1.666	658	1.671	**1.338**
1.2	−1.409		0.944			1.709			1.809	595	1.814	**1.512**
1.3	−1.467		0.974			1.867			1.969	537	1.974	**1.702**
1.4	−1.519		1.000			2.019			2.150	485	2.156	**1.908**
1.5	−1.563		1.024			2.166			2.352	438	2.358	**2.134**
1.6	−1.608		1.045			2.308			2.578	396	2.584	**2.381**
1.7	−1.647		10.65			2.447			2.830	0.357	2.836	**2.652**

$_9X_2 = 0.849$. $b_{2.9} = A_9\,(0.849) = 0.997$, $b_{3.9} = E_9\,(0.849) = 2.004$

Table. 10.

x	$p_{2.10}$	$f_{3.9}$	$p_{3.10}$	A_{10}	B_{10}	G_{10}	E_{10}	D_{10}	$D_{10}(G_{10})$	S_{10}	$u_{2.10}$	$u_{3.10}$
0.0	0.000	−2.006	0.000								**0.990**	**2.004**
0.1	0.005	−1.813	−0.191								**0.995**	**1.813**
0.2	0.020	−1.639	−0.364								**1.010**	**1.640**
0.3	0.046	−1.484	−0.520								**1.036**	**1.484**
0.4	0.082	−1.341	−0.661								**1.072**	**1.343**
0.5	0.129	−1.213	−0.789								**1.119**	**1.215**
0.6	187	−1.097	−0.904								**1.177**	**1.100**
0.7	257	−0.993	−1.009								**1.247**	**0.995**
0.8	340	−0.898	−1.103	1.040	0.790	1.624	1.803	1.003	2.335	0.432	**1.330**	**901**
0.9	436	−0.812	−1.189	0.944	832	1.224	2.089	1.189	1.735	−0.354	**1.426**	**815**
1.0	546	−0.735	−1.266	0.834	871	0.906	2.266	1.366	1.197	−1.069	**1.536**	**738**
1.1	672	−0.665	−1.336		906			1.536			**1.662**	**668**
1.2	815	−0.602	−1.399		937			1.699			**1.805**	**605**
1.3	0.976	−0.544	−1.457		966			1.857			**1.966**	**547**
1.4	1.157	−0.492	−1.509		0.991			2.009			**2.147**	**495**
1.5	1.360	−0.445	−1.555		1.014			2.155			**2.350**	**449**
1.6	1.586	−0.403	−1.598		1.035			2.298			**2.576**	**406**
1.7	1.838	−0.365	−1.636		1.054			2.436			**2.828**	**0.368**

$_{10}X_2 = 0.854$, $b_{2.10} = A_{10}\,(0.854) = 0.990$. $b_{3.10} = E_{10}\,(0.854) = 2.004$

6. Convergency of solutions

Let

$$p_{j,n}(x) = \int_0^x F_j(x,\, u_{1.n-1}, u_{2.n-1},\, u_{3.n-1})\,dx,\quad p_j(x) = \int_0^x F_j(x,\, u_1,\, u_2,\, u_3)\,dx$$

$$j = 1,\ 2,\ 3$$

then we get

$$u_{j,n}(x) = p_{j,n}(x) + b_{j,n}, \quad u_j(x) = p_j(x) + b_j \quad \cdots\cdots (6),$$

and by putting

$$\left.\begin{aligned} u_{j,n}(x) - u_j(x) &= \varepsilon_{j,n}(x), \quad b_{j,n} - b_j = \sigma_{j,n} \\ p_{j,n}(x) - p_j(x) &= \delta_{j,n}(x), \quad {}_nX_j - X_j = \lambda_{j,n} \end{aligned}\right\} \cdots\cdots (7)$$

$$j = 1, 2, 3$$

we have

$$\varepsilon_{j,n}(x) = \delta_{j,n}(x) + \sigma_{j,n} \quad \cdots\cdots (8).$$

Now let the interval of x be $(0, X_0)$ and let $\partial F_j/\partial u_1$, $\partial F_j/\partial u_2$, $\partial F_j/\partial u_3$ be finite at every value of x which laies in this interval, then there exist some fixed positive constants L, M, N which satisfy

$$\left|\frac{\partial F_1}{\partial u_1}\right|, \quad \left|\frac{\partial F_2}{\partial u_1}\right|, \quad \left|\frac{\partial F_3}{\partial u_1}\right| \leqq L$$

$$\left|\frac{\partial F_1}{\partial u_2}\right|, \quad \left|\frac{\partial F_2}{\partial u_2}\right|, \quad \left|\frac{\partial F_3}{\partial u_2}\right| \leqq M$$

$$\left|\frac{\partial F_1}{\partial u_3}\right|, \quad \left|\frac{\partial F_2}{\partial u_3}\right|, \quad \left|\frac{\partial F_3}{\partial u_3}\right| \leqq N$$

uniformly in this interval. Next let $\varepsilon'_{j,n}$ be the greatest value of $|\varepsilon_{j,n}(x)|$ in this interval $(0, X_0)$ and put

$$\rho_n = L\varepsilon'_{1.n} + M\varepsilon'_{2.n} + N\varepsilon'_{3.n} \quad \cdots\cdots (9),$$

then

$$|\delta_{j,n}(x)| = \left|\int_0^x \left[F_j(x, u_{1.n-1}, u_{2.n-1}, u_{3.n-1}) - F_j(x, u_1, u_2, u_3)\right] dx\right|$$

$$\leqq \rho_{n-1} x \leqq \rho_{n-1} X_0$$

that is

$$|\delta_{j,n}(x)| \leqq \rho_{n-1} X_0 \quad \cdots\cdots (10),$$

therefore

$$|\varepsilon_{j,n}(x)| \leqq |\delta_{j,n}(x)| + |\sigma_{j,n}| \leqq \rho_{n-1} x + \sigma'_{j,n}$$

where $\sigma'_{j,n} = |\sigma_{j,n}|$. Moreover let x' be the value of x which satisfies $|\varepsilon_{j,n}(x)| = \varepsilon'_{j,n}$, then

$$\varepsilon'_{j,n} \leqq \rho_{n-1} x' + \sigma'_{j,n} \leqq \rho_{n-1} X_0 + \sigma'_{j,n}$$

that is

$$\varepsilon'_{j,n} \leqq \rho_{n-1} X_0 + \sigma'_{j,n} \quad \cdots\cdots (11).$$
$$j = 1, 2, 3$$

We can consider that L, M, N are positive values, therefore if $\lim_{n\to\infty} \rho_n = 0$, then $\lim_{n\to\infty} \varepsilon_{j,n} = 0$, that is $\lim_{n\to\infty} u_{j,n} = u_j(x)$. Consequently, it is sufficient that we show $\lim_{n\to\infty} \rho_n = 0$ in order to prove the convergence of the solutions.

Case (i)

Conditions $u_1(0) = 0$, $u_1(a_2) = A_2$, $u_2(a_3) = A_3$.

At first

$$b_{1.n} = 0, \quad b_1 = 0, \quad p_{2.n}(a_3) + b_{2.n} = A_3, \quad p_2(a_3) + b_2 = A_3$$

$$\therefore \quad \sigma_{1.n} = 0, \quad \delta_{2.n}(a_3) + \sigma_{2.n} = 0,$$

$$\therefore \quad \sigma'_{1.n} = 0, \quad \sigma'_{2.n} = |\delta_{2.n}(a_3)| \leqq \rho_{n-1} X_0$$

$$\therefore \quad \varepsilon'_{1.n} \leqq \rho_{n-1} X_0, \quad \varepsilon'_{2.n} \leqq 2\rho_{n-1} X_0 \quad \cdots\cdots (i).$$

Next

$$\int_0^{a_2} F_1(x, u_{1.n}, u_{2.n}, u_{3.n})\, dx = A_2, \quad \int_0^{a_2} F_1(x, u_1, u_2, u_3)\, dx = A_2$$

$$\therefore \quad 0 = \int_0^{a_2} \left[F_1(x, u_{1.n}, u_{2.n}, u_{3.n}) - F_1(x, u_1, u_2, u_3) \right] dx$$

$$= \int_0^{a_2} \left[\varepsilon_{1.n}(x) \frac{\partial}{\partial u_1} F_1(x, U_{1.n}, U_{2.n}, U_{3.n}) + \varepsilon_{2.n}(x) \frac{\partial}{\partial u_2} F_1(x, U_{1.n}, U_{2.n}, U_{3.n}) \right.$$

$$\left. + \varepsilon_{3.n}(x) \frac{\partial}{\partial u_3} F_1(x, U_{1.n}, U_{2.n}, U_{3.n}) \right] dx$$

where $U_{j,n}(x)$ are some suitable functions which are between u_j, $u_{j,n}$.

If there exists a positive constant N_0 which satisfies

$$\varepsilon'_{3.n} N_0 \leqq \left| \int_0^{a_2} \varepsilon_{3.n}(x) \frac{\partial}{\partial u_3} F_1(x, U_{1.n}, U_{2.n}, U_{3.n})\, dx \right|$$

independently of n, then we have

$$\varepsilon'_{3.n} N_0 \leqq \varepsilon'_{1.n} L + \varepsilon'_{2.n} M \leqq (L+2M)\,\rho_{n-1} X_0$$

$$\therefore \quad \varepsilon'_{3.n} \leqq \frac{1}{N_0}(L+2M)\,\rho_{n-1} X_0\,,$$

therefore

$$\rho_n \leqq \left\{L + 2M + \frac{N}{N_0}(L+2M)\right\}\rho_{n-1} X_0$$

$$\therefore \quad \rho_n \leqq RX_0\,\rho_{n-1} \leqq (RX_0)^{n-1}\rho_1$$

where

$$R = (L+2M)\left(1 + \frac{N}{N_0}\right).$$

Consequently if $RX_0 < 1$ then $\lim_{n\to\infty} \rho_n = 0$.

Case (ii)

Conditions

$$u_1(0) = 0, \quad u_2(a_2) = K(u_1(a_2)), \quad u_3(a_3) = H(u_1(a_3),\, u_2(a_3)).$$

At the beginning $b_{1.n} = b_1 = 0 \qquad \therefore \ \sigma_{1.n} = 0$.

Next

$$p_{2.n}(a_2) + b_{2.n} = K(p_{1.n}(a_2)), \quad p_{3.n}(a_3) + b_{3.n} = H(p_{1.n}(a_3),\, p_{2.n}(a_3) + b_{2.n})$$

$$p_2(a_2) + b_2 = K(p_1(a_2)), \quad p_3(a_3) + b_3 = H(p_1(a_3),\, p_2(a_3) + b_2),$$

$$\therefore \quad \delta_{2.n}(a_2) + \sigma_{2.n} = \delta_{1.n}(a_2)\frac{\partial}{\partial u_1} K(p_1(a_2) + \theta\delta_{1.n}(a_2)), \quad 0 < \theta < 1$$

$$\delta_{3.n}(a_3) + \sigma_{3.n} = \delta_{1.n}(a_3)\frac{\partial}{\partial u_1} H(P,Q) + [\delta_{2.n}(a_3) + \sigma_{2.n}]\frac{\partial}{\partial u_2} H(P,Q)$$

where P, Q are suitable values which are between $u_1(a_3)$, $u_{1.n}(a_3)$; $u_2(a_3)$, $u_{2.n}(a_3)$ respectively, and if there exist some fixed positive constants A, B, C which satisfy

$$\left|\frac{\partial}{\partial u_1} K\left(p_1(a_2) + \theta\delta_{1.n}(a_2)\right)\right| \leqq A, \quad \left|\frac{\partial}{\partial u_1} H(P,Q)\right| \leqq B,$$

$$\left|\frac{\partial}{\partial u_2} H(P,Q)\right| \leqq C$$

independenlty of n, then

$$\sigma'_{2.n} = |\sigma_{2.n}| \leqq |\delta_{2.n}(a_2)| + A \cdot |\delta_{1.n}(a_2)| \leqq (1+A) X_1 \rho_{n-1}$$

$$\sigma'_{3.n} = |\sigma_{3.n}| \leqq |\delta_{3.n}(a_3)| + B \cdot |\delta_{1.n}(a_3)| + C \cdot [|\delta_{2.n}(a_3)| + \sigma'_{2.n}]$$

$$\leqq X_0\rho_{n-1} + BX_0\rho_{n-1} + CX_0\rho_{n-1} + C(1+A)X_0\rho_{n-1} = \{1 + B + C(2+A)\} X_0\rho_{n-1}$$

therefore

$$\varepsilon'_{1.n} \leqq X_0 \rho_{n-1}$$

$$\varepsilon'_{2.n} \leqq X_0 \rho_{n-1} + \sigma'_{2.n} \leqq (A+2) X_0 \rho_{n-1}$$

$$\varepsilon'_{3.n} \leqq X_0 \rho_{n-1} + \sigma'_{3.n} \leqq \{2 + B + C(A+2)\} X_0 \rho_{n-1}$$

$$\therefore \quad \rho_n = L\varepsilon'_{1.n} + M\varepsilon'_{2.n} + N\varepsilon'_{3.n} \leqq RX_0 \rho_{n-1} \leqq (RX_0)^{n-1} \rho_1$$

where $R = L + (A+2) M + \{2 + B + C(A+2)\} N$.

Consequentry if $RX_0 < 1$, then $\lim_{n\to\infty} \rho_n = 0$.

For example, on the solutions of

$$F_1 = \frac{1}{1+u_1^2}, \quad F_2 = \frac{1}{1+u_1^2+u_2^2}, \quad F_3 = \frac{1}{1+u_2^2+u_3^2}$$

and

$$u_1(0) = 0, \quad K(u_1) = u_1, \quad H(u_1, u_2) = u_1 + u_2$$

we can put

$$L = M = N = \frac{3\sqrt{3}}{8}, \quad A = B = C = 1$$

therefore $R = \frac{3\sqrt{3}}{8}(1+3+6) = \frac{15}{4}\sqrt{3}$, and from $RX_0 < 1$, we have $X_0 < \frac{4}{15\sqrt{3}}$ $\fallingdotseq 0,154$.

Accordingly if we give the values of a_2, a_3 to be less then 0.154, then $\lim_{n\to\infty} u_{j,n}$ converge to definite values. But actually, as ex. 1, in section. 5., shows in spite of $a_2 = 1 > 0.154$, $a_3 = 1.5 > 0.154$ the solutions converge.

Case (iii)

Conditions

$u_1(0) = 0$

$u_3(x) = H_2(x, u_1)$ ····· at the intersection of the solution curve $u = u_2(x)$ and a given curve $u = \Psi_2(x)$

$u_2(x) = H_3(x, u_1)$ ····· at the intersection of the solution curve $u = u_3(x)$ and a given curve $u = \Psi_3(x)$.

At first $\sigma_{1.n} = 0$.

Next

$$p_{2.n}({}_nX_2) + b_{2.n} = \Psi_2({}_nX_2), \quad p_{3.n}({}_nX_2) + b_{3.n} = H_2[{}_nX_2, p_{1.n}({}_nX_2)]$$

$$p_{3.n}({}_nX_3) + b_{3.n} = \Psi_3({}_nX_3), \quad p_{2.n}({}_nX_3) + b_{2.n} = H_3[{}_nX_3, p_{1.n}({}_nX_3)]$$

$$p_2(X_2) + b_2 = \Psi_2(X_2), \qquad p_3(X_2) + b_3 = H_2[X_2, p_1(X_2)]$$

$$p_3(X_3) + b_3 = \Psi_3(X_3), \qquad p_2(X_3) + b_2 = H_3[X_3, p_1(X_3)]$$

$$\therefore \quad \lambda_{2.n}\Psi'_2(X_2 + \theta\lambda_{2.n}) = \sigma_{2.n} + \lambda_{2.n}F_{2.n-1}(X_2 + \theta'\lambda_{2.n}) + \delta_{2.n}(X_2)$$

$$\therefore \quad \lambda_{2.n}\{\Psi'_2(X_2 + \theta\lambda_{2.n}) - F_{2.n-1}(X_2 + \theta'\lambda_{2.n})\} = \sigma_{2.n} + \delta_{2.n}(X_2)$$

$$\lambda_{2.n}F_{3.n-1}(X_2 + \theta''\lambda_{2.n}) + \delta_{3.n}(X_2) + \sigma_{3.n}$$

$$= \lambda_{2.n}\frac{\partial}{\partial x}H_2(X, P) + [\lambda_{2.n}F_{1.n-1}(X_2 + \theta'''\lambda_{2.n}) + \delta_{1.n}(X_2)]\frac{\partial}{\partial u_1}H_2(X, P)$$

$$\therefore \quad \lambda_{2.n}[F_{3.n-1}(X_2 + \theta''\lambda_{2.n}) - \frac{\partial}{\partial x}H_2(X, P) - F_{1.n-1}(X_2 + \theta'''\lambda_{2.n})\frac{\partial}{\partial u_1}H_2(X, P)]$$

$$= -\delta_{3.n}(X_2) - \sigma_{3.n} + \delta_{1.n}(X_2)\frac{\partial}{\partial u_1}H_2(X, P)$$

so that, putting

$$\Psi'_2(X_2 + \theta\lambda_{2.n}) - F_{2.n-1}(X_2 + \theta'\lambda_{2.n}) = A_n, \quad \frac{\partial}{\partial u_1}H_2(X, P) = D_n$$

$$F_{3.n-1}(X_2 + \theta''\lambda_{2.n}) - \frac{\partial}{\partial x}H_2(X, P) - F_{1.n-1}(X_2 + \theta'''\lambda_{2.n})\frac{\partial}{\partial u_1}H_2(X, P) = B_n$$

gives

$$\lambda_{2.n}A_n = \sigma_{2.n} + \delta_{2.n}(X_2), \quad \lambda_{2.n}B_n = \delta_{1.n}(X_2)D_n - \sigma_{3.n} - \delta_{3.n}(X_2)$$

then

$$B_n\sigma_{2.n} + A_n\sigma_{3.n} = A_nD_n\delta_{1.n}(X_2) - A_n\delta_{3.n}(X_2) - B_n\delta_{2.n}(X_2)$$

Similarly, if we put

$$\Psi'_3(X_3 + \theta^{IV}\lambda_{3.n}) - F_{3.n-1}(X_3 + \theta^{V}\lambda_{3.n}) = E_n, \quad \frac{\partial}{\partial u_1}H_3(X', P') = J_n$$

$$F_{2.n-1}(X_3+\theta^{VI}\lambda_{3.n})-\frac{\partial}{\partial x}H_3(X', P')-F_{1.n-1}(X_3+\theta^{VII}\lambda_{3.n})\frac{\partial}{\partial u_1}H_3(X', P')=G_n$$

we get

$$E_n\sigma_{2.n}+G_n\sigma_{3.n}=E_uJ_n\delta_{1.n}(X_3)-E_n\delta_{2.n}(X_3)-G_n\delta_{3.n}(X_3)$$

and

$$\Delta\sigma_{2.n}=\Delta_2\,,\quad \Delta\sigma_{3.n}=\Delta_3$$

where

$$T_n=E_nJ_n\delta_{1.n}(X_3)-E_n\delta_{2.n}(X_3)-\cdot G_n\delta_{3.n}(X_3)$$

$$S_n=A_nD_n\delta_{1.n}(X_2)-A_n\delta_{3.n}(X_2)-B_n\delta_{2.n}(X_2)$$

$$\Delta=B_nG_n-A_nE_n\,,\quad \Delta_2=S_nG_n-T_nA_n\,,\quad \Delta_3=T_nB_n-S_nE_n\,.$$

Moreover if there exist some fixed constants which satisfy following inequalities uniformly

$$0<T_0\leqq|B_nG_n-A_nE_n|,\ |A_n|\leqq A,\ |B_n|\leqq B,\ |D_n|\leqq D,$$

$$|E_n|\leqq E,\ |G_n|\leqq G,\ |J_n|\leqq J$$

we have

$$\sigma'_{2.n}\leqq\frac{1}{T_0}(G\cdot|S_n|+A\cdot|T_n|)$$

$$\leqq\frac{1}{T_0}[ADG+2AG+BG+AEJ+AE]X_0\rho_{n-1}$$

$$\sigma'_{3.n}\leqq\frac{1}{T_0}(B\cdot|T_n|+E\cdot|S_n|)$$

$$\leqq\frac{1}{T_0}[BEJ+2BE+BG+AED+AE]X_0\rho_{n-1}\,.$$

Therefore, by putting

$$\frac{1}{T_0}[ADG+2AG+BG+AEJ+AE]=U$$

$$\frac{1}{T_0}[BEJ+2BE+BG+AED+AE]=V$$

we have

$$\varepsilon'_{1.n}\leqq X_0\rho_{n-1}\,,\quad \varepsilon'_{2.n}\leqq(1+U)X_0\rho_{n-1}\,,\quad \varepsilon'_{3.n}\leqq(1+V)X_0\rho_{n-1}$$

$$\therefore\quad \rho_n=L\varepsilon'_{1.n}+M\varepsilon'_{2.n}+N\varepsilon'_{3.n}\leqq RX_0\rho_{n-1}\leqq(RX_0)^{n-1}\rho_1$$

where

$$R = L + M(1 + U) + N(1 + V).$$

Consequently if $RX_0 < 1$, then $\lim_{n\to\infty} \rho_n = 0$.

For example, on the solutions of ex. 2 in section. 5, we can put

$$L = \frac{3\sqrt{3}}{8\times4\times2} \fallingdotseq 0.082,\quad M = \frac{3\sqrt{3}}{8\times5\sqrt{5}} \fallingdotseq 0.058,\quad N = \frac{3\sqrt{3}}{8\times6\sqrt{6}} \fallingdotseq 0.039$$

and

$$A = \frac{1}{6},\quad B = E = 1,\quad D = J = 0,\quad G = 2,\quad T_0 = \frac{13}{10}$$

$$\therefore\quad U \fallingdotseq 2.18,\quad V \fallingdotseq 3.20$$

then

$$R \fallingdotseq 0.429 \qquad \therefore\quad X_0 < \frac{1}{R} \fallingdotseq 2.33.$$

Consequently, if we can decide the value of X_0 to be $X_0 < 2.33$ by some way, we can assert the convergence of the solutions. Indeed, the calculation shows that $X_0 < 2.33$ and the solutions converge very rapidly as shown in the tables.

7. Conclusion

The method which is discussed here is an extention of the method which we treated of the second order differential equation (Kumamoto J of Sc vol. 1, No. 3, 1954).

The sufficient conditions for the convergence of the solutions which is mentioned in section. 6, are so narrow that they seem worthless in the practical procedure, they are no more than to show a case in which the solutions converge actually, can exist. But it is evident that if $\lim_{n\to\infty} u_{i,n}$, $i = 1, 2, 3$ converge to u_i uniformly, then u_i are required solutions, and in fact as examples show, in most cases $\lim_{n\to\infty} u_{i,n}$ approach to definite values. We want to know more precise conditions for the convergence of the solutions, if possible, the necessary and sufficient conditions, but as far as I know there is no more precise one, and it is our next research.

References

1. Adachi: On the numerical solution of the second order differential equation under some conditions (Kumamoto J of Sc Vol. 1. No. 3)
2. Adachi: On the numerical solution of the simultaneous differential equations under some conditions (Kumamoto J of Sc Vol. 1, No. 4)

A METHOD ON NUMERICAL SOLUTION OF SOME DIFFERENTIAL EQUATION

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(Received August 28, 1954)

1. Introduction

When the existence of the solution of

$$L[x] + f(x, y) = 0 \quad \cdots\cdots \quad (1)$$

under the boundary conditions

$$y(a) = 0, \quad y(b) = 0, \quad a \leqq x \leqq b \quad \cdots\cdots \quad (2),$$

where $L[y] = \frac{d}{dx}\left\{p(x)\frac{dy}{dx}\right\} + q(x)y$, $p(x) \neq 0$, is known, we consider a method concerning the numerical solution of this differential equation.

Let the **Green's** function of the differential equation

$$L[y] = 0 \quad \cdots\cdots \quad (3)$$

under the conditions (2) be $K(x, \xi)$, then if $y_1(x)$, $y_2(x)$ are two independent solutions of (3) and $y_1(a) = 0$, $y_2(b) = 0$, we have

$$\left.\begin{aligned} K(x, \xi) = K^{(1)}(x, \xi) = k y_1(\xi) y_2(x) \quad \cdots\cdots \quad \xi \leqq x \\ K(x, \xi) = K^{(2)}(x, \xi) = k y_1(x) y_2(\xi) \quad \cdots\cdots \quad x \leqq \xi \end{aligned}\right\} \quad \cdots\cdots \quad (4)$$

where $k^{-1} = p(\xi)[y'_1(\xi) y_2(\xi) - y'_2(\xi) y_1(\xi)]$

and $$y(x) = \int_a^b K(x, \xi) f(\xi, y(\xi))\, d\xi \quad \cdots\cdots \quad (5).$$

The integral equation (5) is equivalent to the differential equation (1) under the conditions (2), therefore we consider how to solve (5) numerically.

2. Method on solution

By an analogous method to **Hidaka's** one,[(1)] we can calculate the values, that is, by dividing the interval (a, b) into m equal intervals, and putting $(b-a)/m = h$, $x_i = \xi_i = a + ih$, $i = 0, 1, 2, \cdots\cdots, m$, $y(x_i) = y_i$, $K(x_i, \xi_j) = K_{i,j}$, we get

$$y(x) = \sum_{j=0}^{m} R_j K(x, \xi_j) f(\xi_j, y_j) \quad \cdots\cdots (6),$$

where R_j are some known constants depend on the method of weighted mean which applied. Let $x = x_i$ in (6), then we have

$$y_i = \sum_{j=0}^{m} R_j K_{i,j} f(\xi_j, y_j) \quad \cdots\cdots (7),$$

and putting $i = 0, 1, 2, \cdots\cdots, m$ in this equation, we have $m+1$ simultaneous equatinos with respect to $y_0, y_1, y_2, \cdots\cdots y_m$. Therefore if we can solve these simultaneous equations numerically, the problem is solved.

3. Iterative methhod

In most cases an iterative method is usefull one, that is, starting from any approximate values $y_{i,1}$, calculate the values of $y_{i,2}, y_{i,3}, \cdots\cdots$ successively by the relations

$$y_{i,n} = \sum_{j=0}^{m} R_j K_{i,j} f(\xi_j, y_{j,n-1}) \quad \cdots\cdots (8)$$

$$i = 0, 1, 2, \cdots\cdots, m$$

In this case, if we put $\varepsilon_{i,n} = y_{i,n} - y_i$, we have

$$\varepsilon_{i,n} = \sum_{j=0}^{m} R_j K_{i,j} \left\{ f(\xi_j, y_{j,n-1}) - f(\xi_j, y_j) \right\}$$

$$= \sum_{j=0}^{m} R_j K_{i,j} \varepsilon_{j,n-1} f_y(\xi_j, y_j + \theta \varepsilon_{j,n-1}) \quad \cdots\cdots (9)$$

$$0 < \theta < 1$$

Therefore, if there exist some positive constants F_j, K_j which satisfy

$$|f_y(\xi_j, y)| \leqq F_j, \quad |K_{i,j}| = K_j$$

we get $|\varepsilon_{i,n}| \leqq \rho_{n-1}$ where $|R_j K_j F_j| = G_j$, $\sum_{j=0}^{m} G_j |\varepsilon_{j,n-1}| = \rho_{n-1}$.

Consequently we have

$$\rho_n = \sum_{j=0}^{m} G_j |\varepsilon_{j,n}| \leqq \sum_{j=0}^{m} G_j \rho_{n-1} = G \rho_{n-1}, \quad \text{where } \sum_{j=0}^{m} G_j = G.$$

therefore $\rho_n \leqq G \rho_{n-1}$, that is $\rho_n \leqq G^{n-1} \rho_1$ $\cdots\cdots (10)$.

Accordingly if $G<1$, then we get

$$\lim_{n\to\infty} \rho_n = 0, \quad \text{that is,} \quad \lim_{n\to\infty} y_{i,n} = y_i$$
$$i = 0, 1, 2, \cdots\cdots\cdots, m.$$

Of course, we must consider not only above errors but also the errors which appear by the substitution of (6) in place of (5).

4. Example

We can easily show that **non-linear differential equation**

$$\frac{d^2y}{dx^2} + \alpha y - \beta y^3 = 0 \quad \cdots\cdots\cdots (11)$$

where α, β are positive constants, can have a periodic solution whose period T is any value not less than $2\pi/\sqrt{\alpha}$.

As an example we put $\alpha = 12$, $\beta = 10$, $T = 2$, $(2\pi/\sqrt{12} < 2)$, then by the symmetry of the solution, we can put $y(0) = 0$, $y(1) = 0$ and consider the interval $0 \leqq x \leqq 1$. In this case

$$K^{(1)}(x, \xi) = \xi(1-x), \quad K^{(2)}(x, \xi) = x(1-\xi) \quad \cdots\cdots\cdots (12),$$

and

$$y(x) = \int_0^1 K(x, \xi)\,[12\,y(\xi) - 10\,y^3(\xi)]\,d\xi \quad \cdots\cdots\cdots (13).$$

Put $m=20$, and by applying **Simpson's** $\frac{1}{3}$- and $\frac{3}{8}$ **rules** and considering $y_i = y_{20-i}$, $(i = 1, 2, \cdots\cdots, 9)$, $y_0 = y_{20} = 0$, we have

$$y_i = \frac{1}{30}\,[2(K^{(1)}_{i,1} + K^{(2)}_{i,19})\,Z_1 + (K^{(1)}_{i,2} + K^{(2)}_{i,18})\,Z_2 + 2(K^*_{i,3} + K^{(2)}_{i,17})\,Z_3$$
$$+ (K^*_{i,4} + K^{(2)}_{i,16})\,Z_4 + 2(K^*_{i,5} + K^{(2)}_{i,15})\,Z_5 + (K^*_{i,6} + K^{(2)}_{i,14})\,Z_6 + 2(K^*_{i,7}$$
$$+ K^{(2)}_{i,13})\,Z_7 + (K^*_{i,8} + K^{(2)}_{i,12})\,Z_8 + 2(K^*_{i,9} + K^{(2)}_{i,11})\,Z_9 + K^{(2)}_{i,10}\,Z_{10}] \quad \cdots\cdots (14)$$
$$i = 2, 4, 6, 8, 10$$

$$y_i = \frac{9}{160}(K^{(1)}_{i,1} + K^{(2)}_{i,20})\,Z_1 + \frac{9}{160}(K^{(1)}_{i,2} + K^{(2)}_{i,18})\,Z_2 + \frac{17}{480}(K^{(1)}_{i,3} + K^{(2)}_{i,17})\,Z_3$$
$$+ \frac{1}{30}\,[2(K^*_{i,4} + K^{(2)}_{i,16})\,Z_4 + (K^*_{i,5} + K^{(2)}_{i,15})\,Z_5 + 2(K^*_{i,6} + K^{(2)}_{i,14})\,Z_6$$

$$+ (K^{*}_{i,7} + K^{(2)}_{i,13}) Z_7 + 2(K^{*}_{i,8} + K^{(2)}_{i,12}) Z_8 + (K^{(2)}_{i,9} + K^{(2)}_{i,11}) Z_9 + 2K^{(2)}_{i,10} Z_{10}] \cdots (15)$$

$$i = 3, 5, 7, 9$$

$$y_1 = \frac{9}{160}\left(K^{(1)}_{1,1} + K^{(2)}_{1,19}\right)Z_1 + \left(\frac{1}{15} K^{(2)}_{1,2} - \frac{1}{96} K^{(1)}_{1,2} + \frac{9}{160} K^{(2)}_{1,18}\right) Z_2$$
$$+ \left(\frac{1}{480} K^{(1)}_{1,3} + \frac{1}{30} K^{(2)}_{1,3} + \frac{17}{480} K^{(2)}_{1,17}\right)Z_3 + \frac{1}{30}\left[2\left(K^{(2)}_{1,4} + K^{(2)}_{1,16}\right) Z_4\right.$$
$$+ \left(K^{(2)}_{1,5} + K^{(2)}_{1,15}\right) Z_5 + 2\left(K^{(2)}_{1,6} + K^{(2)}_{1,14}\right) Z_6 + \left(K^{(2)}_{1,7} + K^{(2)}_{1,13}\right) Z_7$$
$$\left. + 2\left(K^{(2)}_{1,8} + K^{(2)}_{1,12}\right) Z_8 + \left(K^{(2)}_{1,9} + K^{(2)}_{1,11}\right) Z_9 + 2K^{(2)}_{1,10} Z_{10}\right] \cdots\cdots (16)$$

where

$$Z_i = 12 y_i - 10 y_i^3 \cdots\cdots (17),$$

and $K^{*}_{i,j} = K^{(1)}_{i,j}$ when $i \geqq j$, $K^{*}_{i,j} = K^{(2)}_{i,j}$ when $i \leqq j$,

Substituting

$$K^{(1)}_{i,j} = \frac{1}{400}(20 - i) j, \quad K^{(2)}_{i,j} = \frac{1}{400} i (20 - j)$$

gives

$$y_1 = \frac{1}{400}\left[\frac{9}{8} Z_1 + \frac{11}{12} Z_2 + \frac{19}{24} Z_3 + \frac{4}{3} Z_4 + \frac{2}{3} Z_5 + \frac{4}{3} Z_6\right.$$
$$\left. + \frac{2}{3} Z_7 + \frac{4}{3} Z_8 + \frac{2}{3} Z_9 + \frac{2}{3} Z_{10}\right]$$
$$y_2 = \frac{1}{400}\left[\frac{4}{3} Z_1 + \frac{4}{3} Z_2 + \frac{8}{3} Z_3 + \frac{4}{3} Z_4 + \frac{8}{3} Z_5 + \frac{4}{3} Z_6\right.$$
$$\left. + \frac{8}{3} Z_7 + \frac{4}{3} Z_8 + \frac{8}{3} Z_9 + \frac{2}{3} Z_{10}\right]$$
$$y_3 = \frac{1}{400}\left[\frac{9}{8} Z_1 + \frac{9}{4} Z_2 + \frac{17}{8} Z_3 + 4 Z_4 + 2 Z_5 + 4 Z_6\right.$$
$$\left. + 2 Z_7 + 4 Z_8 + 2 Z_9 + 2 Z_{10}\right]$$
$$y_4 = \frac{1}{400}\left[\frac{4}{3} Z_1 + \frac{4}{3} Z_2 + 4 Z_3 + \frac{8}{3} Z_4 + \frac{16}{3} Z_5 + \frac{8}{3} Z_6\right.$$
$$\left. + \frac{16}{3} Z_7 + \frac{8}{3} Z_8 + \frac{16}{3} Z_9 + \frac{4}{3} Z_{10}\right]$$
$$y_5 = \frac{1}{400}\left[\frac{9}{8} Z_1 + \frac{9}{4} Z_2 + \frac{17}{8} Z_3 + \frac{16}{3} Z_4 + \frac{10}{3} Z_5\right.$$
$$\left. + \frac{20}{3} Z_6 + \frac{10}{3} Z_7 + \frac{20}{3} Z_8 + \frac{10}{3} Z_9 + \frac{10}{3} Z_{10}\right] \cdots\cdots (18)$$

$$y_6 = \frac{1}{400}\left[\frac{4}{3}Z_1 + \frac{4}{3}Z_2 + 4Z_3 + \frac{8}{3}Z_4 + \frac{20}{3}Z_5 + 4Z_6 + 8Z_7 + 4Z_8 + 8Z_9 + 2Z_{10}\right]$$

$$y_7 = \frac{1}{400}\left[\frac{9}{8}Z_1 + \frac{9}{4}Z_2 + \frac{17}{8}Z_3 + \frac{16}{3}Z_4 + \frac{10}{3}Z_5 + 8Z_6 + \frac{14}{3}Z_7 + \frac{28}{3}Z_8 + \frac{14}{3}Z_9 + \frac{14}{3}Z_{10}\right]$$

$$y_8 = \frac{1}{400}\left[\frac{4}{3}Z_1 + \frac{4}{3}Z_2 + 4Z_3 + \frac{8}{3}Z_4 + \frac{20}{3}Z_5 + 4Z_6 + \frac{28}{3}Z_7 + \frac{16}{3}Z_8 + \frac{32}{3}Z_9 + \frac{8}{3}Z_{10}\right]$$

$$y_9 = \frac{1}{400}\left[\frac{9}{8}Z_1 + \frac{9}{4}Z_2 + \frac{17}{8}Z_3 + \frac{16}{3}Z_4 + \frac{10}{3}Z_5 + 8Z_6 + \frac{14}{3}Z_7 + \frac{32}{3}Z_8 + 6Z_9 + 6Z_{10}\right]$$

$$y_{10} = \frac{1}{400}\left[\frac{4}{3}Z_1 + \frac{4}{3}Z_2 + 4Z_3 + \frac{8}{3}Z_4 + \frac{20}{3}Z_5 + 4Z_6 + \frac{28}{3}Z_7 + \frac{16}{3}Z_8 + 12Z_9 + \frac{10}{3}Z_{10}\right]$$

Re-arranging (18), gives

$$\begin{aligned} y_8 &= y_{10} - u_9, & y_7 &= y_9 - u_8 \\ y_6 &= 2y_8 - y_{10} - u_7, & y_5 &= 2y_7 - y_9 - u_6 \\ y_4 &= 2y_6 - y_8 - u_5, & y_3 &= 2y_5 - y_7 - u_4 \\ y_2 &= 2y_4 - y_6 - u_3, & y_1 &= 2y_3 - y_5 - u_2 \end{aligned} \quad \cdots\cdots (19)$$

where

$$u_9 = \frac{1}{300}\left(Z_9 + \frac{1}{2}Z_{10}\right), \quad u_i = \frac{1}{300}\left(Z_i + Z_{i+1} + Z_{i+2}\right) \quad \cdots\cdots (20).$$
$$i = 2, 3, \cdots\cdots, 8$$

At first, find the values fo $y_{10.n}$, $y_{9.n}$ from

$$y_{10.n} = \frac{1}{400}\left[\frac{4}{3}Z_{1.n-1} + \frac{4}{3}Z_{2.n-1} + 4Z_{3.n-1} + \frac{8}{3}Z_{4.n-1} + \frac{20}{3}Z_{5.n-1} + 4Z_{6.n-1} + \frac{28}{3}Z_{7.n-1} + \frac{16}{3}Z_{8.n-1} + 12Z_{9.n-1} + \frac{10}{3}Z_{10.n-1}\right]$$

$$y_{9.n} = \frac{1}{400}\left[\frac{9}{8}Z_{1.n-1} + \frac{9}{4}Z_{2.n-1} + \frac{17}{8}Z_{3.n-1} + \frac{16}{3}Z_{4.n-1} + \frac{10}{3}Z_{5.n-1} + 8Z_{6.n-1} + \frac{14}{3}Z_{7.n-1} + \frac{32}{3}Z_{8.n-1} + 6Z_{9.n-1} + 6Z_{10.n-1}\right]$$

then calculate the values of $y_{i,n}$ $(i = 8,7,6,5,4,3,2,1)$from

$$y_{8.n} = y_{10.n} - u_{9.n-1}, \qquad y_{7.n} = y_{9.n} - u_{8.n-1}$$

$$y_{6.n} = 2y_{8.n} - y_{10.n} - u_{7.n-1}, \qquad y_{5.n} = 2y_{7.n} - y_{9.n} - u_{6.n-1}$$

$$y_{4.n} = 2y_{6.n} - y_{8.n} - u_{5.n-1}, \qquad y_{3.n} = 2y_{5.n} - y_{7.n} - u_{4.n-1}$$

$$y_{2.n} = 2y_{4.n} - y_{6.n} - u_{3.n-1}, \qquad y_{1.n} = 2y_{3.n} - y_{5.n} - u_{2.n-1}$$

successively where $u_{i,n-1}$ are easily obtained by mental arithmetic from the tables of the values of $Z_{i,n-1}$. In the table.1, $y_{i,n}$ are the values which are obtained by this method starting from $y_{0.1} = 0$, $y_{2.1} = 0.2$, $y_{4.1} = 0.4$, $y_{6.1} = 0.6$, $y_{8.1} = 0.8$, $y_{10.1} = 0.9$.

Table. 1.

i	$yi,_1$	·····	$yi,_2$	····	$yi,_3$	$y'i,_3$	$Zi,_3$				
0	0.0	·····	0,000	····	0.000	0.00	0.000				
1						0.08	0.959	9/8···1.08	4/3···1.28	$y_{10.4}=0.5268$ $u_{9.3}=0.0242$	$u_{7.3}=0.0473$ $y_{10.4}=0.5268$
2	0.2		0.160		0.161	0.16	1.879	9/4···4.23	4/3···2.51	$y_{8.4}=0.5026$	0.5741
3						24	2.74	17/8···5.82	4···10.96		$2y_{8.4}=1.0052$ 0.5741
4	0.4		0.314		0.317	32	3.51	16/3·· 18.72	8/3···9.39		$y_{6.4}=0.4311$
5		·····		····		38	4.01	10/3·· 13.37	20/3·· 26.73		
6	0.6	·····	0.442	····	0.429	43	4.36	8···34.88	4···17.44	$y_{9.4}=0.5211$	$y_{9.4}=0.5211$
7						47	4.60	14/3·· 21.47	28/3·· 42.93	$u_{8.3}=0.0482$	$u_{6.3}=0.0457$
8	0.8		0.487		0.504	50	4.75	32/3·· 50.67	16/3·· 25.33	$y_{7.4}=0.4729$	0.5668
9						52	4.83	6···28.98	12···57.96		$2y_{7.4}=0.9458$ 0.5668
10	0.9	·····	0.525	····	0.525	0.53	4.87	6···29.22	10/3·· 16.23		$y_{5.4}=0.3790$
								208.44 $y_{9.4}=0.5211$	210.73 $y_{10.4}=0.5268$		

i			$yi,_4$	····	$yi,_7$	·····	$yi,_9$	·····	$yi,_{10}$	Yi	$\bar{y}i,_9$
0	$y_{8.4}=0.5026$ $u_{5.3}=0.0432$	$y_{6.4}=0.4311$ $u_{3.3}=0.0342$	0.0000	····	0.0000	·····	0.00000	·····	0.00000	0.0000	0.0000
1	0.5458	0.4653	0.0849		0.0859		0.08583		0.08590	0.0859	
2	$2y_{6.4}=0.8622$	$2y_{4.1}=0.6328$	0.1675		0.1690		0.16918		0.16922	0.1692	0.1624
3	0.5458	0.4653	2455		2473		24755		24764	2476	
4	$y_{4.4}=0.3164$	$y_{2.4}=0.1675$	3164		3186		31896		31902	3190	0.3171
5			3790	····	3812	·····	38160	·····	38170	3817	
6	$y_{7.4}=0.4729$ $u_{4.3}=0.0396$	$y_{5.4}=0.3790$ $u_{2.3}=0.0271$	4311	····	4338	·····	43427	·····	43434	4342	0.4323
7	0.5125	0.4061	4729		4755		47593		47603	4759	
8	$2y_{5.4}=0.7580$	$2y_{3.4}=0.4910$	5026		5056		50607		50616	5061	0.5039
9	0.5125	0.4061	5211		5238		52426		52436	5243	
10	$y_{3.4}=0.2455$	$y_{1.4}=0.0849$	0.5268	····	0.5298	·····	0.53036	·····	0.53045	0.5304	0.5283

4. Some comments

In the table. 1, Y_i are the values which were obtained by **Adachi**'s method[2] which is quite different method applicable in more general cases, and y_i , Y_i nearly coinside with each other. $\dot{y}_i$ are the values which were obtained by the method explained above putting $m = 10$ starting from the same initial values, and show that the errors which were arised by the substitution of (6) in place of (5) are considerably large. It is well known result that the upper limit of the errors of **Simpson**'s rule is proportional to $(b-a)^5/m^4$. Therefore it is obvious that we have good result when m is large. But above example is rather bad example and in most cases it is sufficient to take $m = 10$.

References

(1) Hidaka: Applied integral equation §52 (in japanese)
(2) Adachi: On the numerical solution of the second order differential equation under some conditions. Kumamoto J of Sc Vol, 1. No. 3 (1954)

AN EXPERIMENT ON THE TEMPERATURE ERROR OF A PENDULUM CLOCK

Ryuzo ADACHI

(Received August 28, 1954)

1. Introduction

If Δ is the error of a time keeper at any time per unit time, then the total error τ during t is given

$$\tau = \int_{To}^{To+t} \Delta \, dt \qquad \cdots\cdots (1)$$

where Δ may be a function of temperature, atmospheric pressure, gravity etc., but in the following, we consider the temperature error which is generally greatest.

Now, put

$$\Delta = a_0 + a_1 \theta \qquad \cdots\cdots (2)$$

where θ is temperature, a_0, a_1 are some constants, and consider how to determine the values of a_0, a_1. From (1), (2) we get

$$\tau = a_0 t + a_1 \int_{To}^{To+t} \theta \, dt \qquad \cdots\cdots (3).$$

Let the curve $\theta = f(t)$ be obtained by using a self-recording thermometer, then the values of

$$A_{m.i} = \int_{Ti-1}^{Ti-1+Ti} \theta^m \, dt \qquad \text{and} \quad \tau_i = \int_{Ti-1}^{Ti-1+Ti} \Delta \, dt \qquad \cdots\cdots (4)$$

$$m = 0, 1; \quad i = 1, 2, \cdots\cdots, p; \quad p > m$$

can be obtained experimentally, and we have following observational equations

$$\left.\begin{aligned} &A_{0.1}\, a_0 + A_{1.1}\, a_1 = \tau_1 \\ &A_{0.2}\, a_0 + A_{1.2}\, a_1 = \tau_2 \\ &\cdots\cdots\cdots\cdots \\ &A_{0.i}\, a_0 + A_{1.i}\, a_1 = \tau_i \\ &\cdots\cdots\cdots\cdots \\ &A_{0.p}\, a_0 + A_{1.p}\, a_1 = \tau_p \end{aligned}\right\} \qquad \cdots\cdots (5).$$

Solving these equations with respect to a_0, a_1 by the method of least square, we can obtain their values.

2. Determination of $\int_\beta^\alpha \theta\, dt$

In most cases, the self-recording thermometer which we use is cylinder-drum type and the graph is as shown in Fig. 3, its abscissa is straight line and the ordinate is arc of a circle.

Generally, if $\varphi(\Theta)$ is a one valued function of Θ and we consider a transformation $(t, \theta) \leftrightarrows (T, \Theta)$ by the relation

$$t = T - \varphi(\Theta), \quad \theta = \Theta \qquad \cdots\cdots (6),$$

it means that the intersection of the curve $T = \varphi(\Theta) + const = \varphi(\Theta) + C_1$, and $\Theta = const = C_2$ in the (T, Θ) plane is transformed to the intersection of two straight lines $t = const = C_1$, $\theta = const\ t = C_2$ in the (t, θ) plane. Therefore in (T, Θ) plane a domain G which is bounded by four lines $\Theta = 0$, $T = \varphi(\Theta) + \alpha$, $T = \varphi(\Theta) + \beta$, $T = f(\Theta) + \varphi(\Theta)$ is transformed into $\mathcal{G}$ which is bounded by four lines $\theta = 0$, $t = \alpha$, $t = \beta$, $t = f(\theta)$ in the (t, θ) plane, and if S, s are the areas of G, $\mathcal{G}$ respectively, we have

$$S = \iint_G dT d\Theta, \quad s = \iint_{\mathcal{G}} dt d\theta = \iint_G \frac{\partial(t, \theta)}{\partial(T, \Theta)}\, dT d\Theta$$

while

$$\frac{\partial t}{\partial T} = 1, \quad \frac{\partial t}{\partial \Theta} = -\varphi'(\Theta), \quad \frac{\partial \theta}{\partial T} = 0, \quad \frac{\partial \theta}{\partial \Theta} = 1$$

$$\therefore \quad \frac{\partial(t, \theta)}{\partial(T, \Theta)} = 1 \qquad \therefore \quad s = S$$

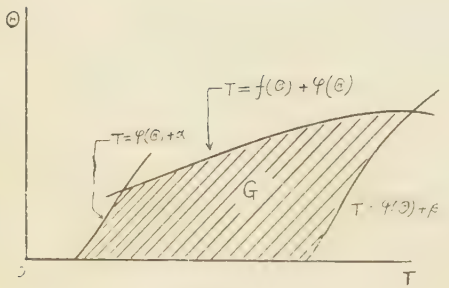

Fig. 1.

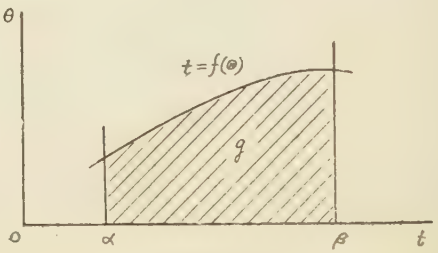

Fig. 2.

Concerning above thermometer

$t = T - a + \sqrt{R^2 - (\Theta - b)^2}$, $\theta \fallingdotseq \Theta$ where R is radius of circle and a, b are some constants, therefore $\varphi(\Theta) = a - \sqrt{R^2 - (\Theta - b)^2}$. Consequently the area of the graph which is recorded is equal to the value of $\int \theta dt$. We have experimented several time keepers, and in the following, show one of them.

3. Experimental result

A pendulum clock was tested, its pendulum is gridiron pendulum and driving force is that of falling weight. The time was compared with **JJY**−signal and the data are as follows.

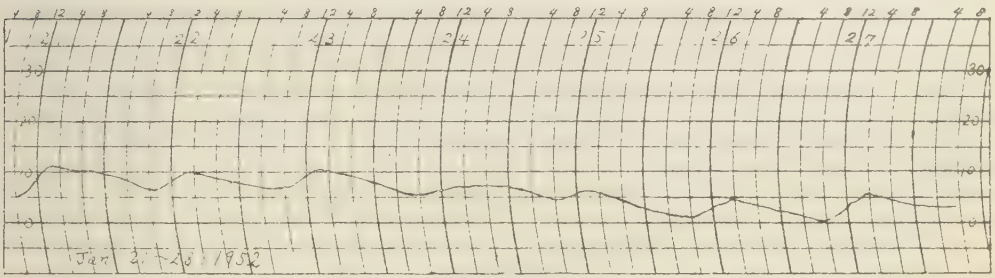

Fig. 3.

From these data, observational equations are

$$\left.\begin{aligned} 24a_0 + 218a_1 &= 4.3k \\ 24a_0 + 198a_1 &= 4.5k \\ 24a_0 + 192a_1 &= 4.6k \\ 24a_0 + 154a_1 &= 5.1k \\ 24a_0 + 80a_1 &= 6.0k \\ 24a_0 + 62a_1 &= 6.2k \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (7)$$

Table. 1.

at noon	error (sec)	τ(sec)	$\int \theta dt$ (deg, hour)
21d	−10.5		
22	−6.2	+4.3	218
23	−1.7	+4.5	198
24	+2.9	+4.6	192
25	+8.0	+5.1	154
26	+14.0	+6.0	80
27	+20.2	+6.2	62

where the units of a_0, a_1 are $hour^{-1}$, $(deg.\ hour)^{-1}$ respectively and $k = 60^{-1}$. From (7) we have as normal equations

$$\left.\begin{aligned} 3456a_0 + 21700a_1 &= 736.8k \\ 21700a_0 + 157600a_1 &= 4361k \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (8)$$

and solving these, we get

$$a_0 = 0.291k, \quad a_1 = -0.0124k \quad .$$

Therefore

$$\tau = [0.291\,t - 0.0124 \int_0^t \theta dt]\ sec \quad \cdots\cdots (9),$$

where t, θ are measured by the units *hour*, *degree. C* respectively.

4. Conclusion

The error except the constant error a_0 is caused mainly by temperature change, atmospheric pressure change etc., but when the time keeper is settled in a definite place and driving force is uniform, the temperature error is very large as compared with others. If the pendulum is perfectly compensated, a_1 must be zero, but in most cases, concerning so-called compensated pendulum clock, a_1 is not zero as above example shows. I have calculated the condition that perfectly $a_1 \equiv 0$ is satisfied and its result is very complex.

References

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QUASI-STATIONARY PROCESS (I). ON ENTROPY PRODUCTION

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(Received Sept. 30 1954)

Abstract

It is proved and explained by Prigogine[1], de Groot[2] and Meixner[3] that the entropy production of irreversible process is minimum in the stationary state. The writer studied the conditions under which the entropy production of irreversible process is minimum in the stationary state, and could acknowledge that there are some cases where the entropy production is considered not minimum in the stationary state. Also he introduced a formula as for the rate of change of forces in irreversible process and thereby he deduced the expressions for the time variations of forces, fluxes and entropy production in non-stationary processes.

1. Introduction

The entropy production of irreversible process becomes constant in the stationary state. Prigogine and de Groot made clear that there are various stationary states which are said to be the 0th, the 1st, the 2nd, ···· order and in these stationary states the entropy production is minimum under the conditions of some fixed forces.

This result was deduced under the assumptions that during the irreversible process some forces are kept constant througout strictly. But it is considered difficult to keep the forces constant in a non-stationary process. As the forces in the system have intimate relations to the state of the system, the change of the state are accompanied by the changes of the forces. Even if some fixed forces are applied to the system, the changes of fluxes produced by the forces will induce some new forces in the direction opposite to that of the changes, and the actual forces which practically exert in the system will be changed in some manner. If we would accomodate the applied forces so that the resultant forces are kept constant always, we could be able to keep the actual forces constant. But this accomodation is very difficult, almost impossible, to realize. In actual irreversible process, therefore, it is considered very difficult to keep any force at constant value except in the case of stationary process.

As the minimum condition of the entropy production in stationary process is obtained under the condition of constancy of some forces, in practical cases in which the actual forces can not be kept constant even if some constant forces are applied on the system, the entropy production may not be minimum necessarily.

We will study the entropy productins in stationary states, and the time variations of forces, fluxes and entropy productions during the changes from non-stationary states to stationary states. In the next paragraph there are stated some simple examples which will

be seen as the cases where the entropy productions ate not minimum in the stationary states.

2. Simple Examples of Stationary States in which the Entropy Productions are not Minimum.

a) When an electric circuit is exerted by a fixed E.M.F., some new E.M.F. is induced in the direction opposite to the increase of the electric current, and the current increases from 0 to a final stationary value which is looked upon as rather maximum than minimum.
b) After the electric current in a circuit becomes stationary under a constant E.M.F., if any magnetic perturbation is given in the manner that the induced E.M.F. in the circuit has the direction opposite to the initial current, the entropy production in the perturbed state becomes smaller and shortly afterwards recover to the initial value.
c) Whern the both ends of a homogeneous straight rod which has uniform section are kept at two constant temperatures T_1 and T_2 $(T_1 > T_2)$ respectively, heat flows from T_1 end to T_2 end, and if heat do not pass through lateral surface, the temperature T of the middle point of the rod is $\frac{T_1+T_2}{2}$ in the stationary flow. But it is proved as follows that the entropy production is minimum in the case of $T=\sqrt{T_1 T_2}$ and not minimum in the case of $\frac{T_1=T_2}{2}$.

If the sectional area, the length and the heat conductivity are denoted by A, $2l$ and k, the entropy production σ caused by the conduction of heat is expressed as following.

$$\sigma = \frac{kA}{l}\left\{\frac{(T_1-T)^2}{TT_1} + \frac{(T-T_2)^2}{TT_2}\right\} \quad \cdots\cdots (1)$$

Differentiating this with T, we get

$$\frac{d\sigma}{dT} = \frac{kA}{l}\left(1-\frac{T_1T_2}{T^2}\right)\left(\frac{1}{T_1}+\frac{1}{T_2}\right) \quad \cdots\cdots (2)$$

$$\frac{d^2\sigma}{dT^2} = \frac{2kA}{l}\frac{T_1T_2}{T^3}\left(\frac{1}{T_1}+\frac{1}{T_2}\right) \quad \cdots\cdots (3)$$

Thus when T is $\sqrt{T_1T_2}$, σ is minimum, but the flow of heat is not stationary, and therefore, if the initial value of the temperature T of the middle point is $\sqrt{T_1T_2}$, T rises gradually to the final stationary value $\frac{T_1+T_2}{2}$, and σ increases. If the initial value of T is higher than $\frac{T_1+T_2}{2}$, T falls gradually to the final stationary value and σ decreases. Thus the value of σ in the stationary state is not minimum nor maximum.

3. Reconsideration of the Theoretical Method which obtained the Result that the Entropy Production in Stationary State is Minimum

Prigogine and de Groot proved the following theorem: when a system, characterized

by n independent forces $X_1, X_2, \cdots\cdots X_n$, is kept in a state with fixed $X_1, X_2, \cdots\cdots X_k$ (k is one of the numbers $1, 2, \cdots\cdots, n$) and the minimum entropy production σ, the fluxes J_i ($i=k+1, k+2, \cdots\cdots, n$) vanish and the state is stationary.

This theorem is proved simply in the following manner. Among forces X_i ($i=1, 2, \cdots\cdots, n$), the fluxes J_i and the entropy production σ, there exist the relations

$$J_i = \sum_{j=1}^{n} L_{ij} X_j \quad (i=1, 2, \cdots\cdots, n) \cdots\cdots (4)$$

$$L_{ij} = L_{ji} \quad (i, j=1, 2, \cdots\cdots, n) \cdots\cdots (5)$$

(Onsager's reciprocal relations)[4],[5]

$$\sigma = \sum_{i=1}^{n} J_i X_i = \sum_{i, j=1}^{n} L_{ij} X_i X_j \cdots\cdots (6)$$

When the values of the forces $X_1, X_2, \cdots\cdots X_k$ are fixed, the state of minimum σ is found from the conditions

$$\left(\frac{\partial \sigma}{\partial X_i}\right) = 0 \quad (i=k+1, k+2, \cdots\cdots, n) \cdots\cdots (7)$$

where during the differentiation all the other forces than X_i have to be kept constant. From these conditions, we get

$$\sum_{j=1}^{n} (L_{ij}+L_{ji}) X_j = 0 \quad (i=k+1, k+2, \cdots\cdots, n)$$

or

$$J_i = 0 \quad (i=k+1, k+2, \cdots\cdots, n) \cdots\cdots (8)$$

We now apply a perturbation δX_m (m is one of the numbers $k+1, k+2, \cdots\cdots, n$) to one of the forces, which were originally not fixed, we have a flux $J_m = L_{mm} \delta X_m$. It is known that the positive flux J_m gives rise to a negative change in δX_m, thus the value of δX_m decreases gradually to 0, and the state of minimum entropy production means stationary state.

This proof is correct, but when we take actual irreversible process, we can not consider it possible without difficulty to give the perturbation δX_m to the system without changes of forces other than X_m. δX_m produces flux J_m which is intimately related to all forces, and therefore J_m gives effects to all forces. Thus the forces other than X_m would change if there were no extremely special complex mechanism for keeping constancy of the forces. But the mechanism like this would be almost impossible to realize. Thus, in actual cases, the perturbation which gives δX_m to X_m only and no effect to forces other than X_m would be generally extremely difficult to realize.

We will consider the simple cases in the following.

(a) If any temperature difference is given to two junctions of a thermo-couple, some E.M.F. will be produced. In this case it is difficult to change the temperature difference without changing the E.M.F. It may be possible to change the E.M.F. from outside without changing the temperature difference if we use two heat reservoirs, one of which keeps the temperature of one junction, the other keeps the temperature of the other junction. But it is very difficult to change the E.M.F. without changing the temperature gradient in the couple, since the production and the absorption of heat by Thomson effect in each part of the couple would be changed.

(b) In an electric circuit which is acted by some electric field and some magnetic field at the same time, it is difficult to change electric field without changing magnetic field, also it is difficult to change magnetic field without changing electric field.

(c) In the phenomena of thermal diffusion (Soret effect) and Dufour effect, concentration (temperature) gradient is produced by temperature (concentration) gradient. In this case, it is difficult to change concentration (temperature) gradient without changing temperature (concentration) gradient.

In our opinion, the meaning of that the forces X_1, X_2, ……, X_n are independent, is not that each of the forces is changeable perfectly without giving any effect to other forces, but that the state of the system is defined by giving definite independent values to the forces X_1, X_2, ……, X_n. The change of one of the independent forces may have some effects on the other forces. But these effects can not be determined (if the effects might be determined definitely, the forces are not independent each other). To determine the effects definitely it is necessary to give some more proper conditions. Under different conditions, different effects would be produced.

For example, we would take a definite quantity of gas whose state is defined by two independent variables, say temperature and pressure. If a sudden change of pressure (temperature) were applied, any change of temperature (pressure) would be produced, but the quantity of change of the latter do not be determined without giving further conditions. If we put the gas in a heat reservoir which keeps a constant temperature, and suddenly change the pressure of the gas, the temperature will change instantaneously but soon return to that of the reservoir by conduction, convection and radiation of heat. Also, if we connect the gas to a manostat by a tube, and suddenly change the temperature, the pressure will change instantaneously but soon return to the pressure of manostat. Thus it is very difficult in actual cases to change the temperature (pressure) without giving any change of the pressure (temperature).

We are concerned with the state of the system at the time soon after the application of δX_m. After the application of δX_m, if we keep the constant value of δX_m for a time, the forces other than X_m will soon return to the initial constant values. But as the value of δX_m will change rapidly, the values of the forces other than X_m will change generally

with δX_m. These values of forces will return to the initial values with δX_m at the same time.

From the reasons mentioned above, it is considered almost impossible to give the perturbation δX_m with keeping the constancy of the other forces than X_m. If we use some extremely complex and exquisite mechanism, it may be possible, but in this case, it will be necessary to give any change of the mechanism so as to keep the constancy of forces other than X_m, and the change of the mechanism would have some effect to entropy production.

If any one of the forces $X_1, X_2, \cdots\cdots, X_n$ would change, all the fluxes would change, and as the fluxes have intimate relations to the forces, it is considered right that all the froces would accept some changes.

4. Transformation of the Relations between Forces and Fluxes in Irreversible Processes to Diagonal Forms

When any irreversible process proceeds in a system, if we denote the forces by $X_1, X_2, \cdots\cdots, X_n$, and the conjugate fluxes by $J_1, J_2, \cdots\cdots, J_n$, there exist generally the relations (4).

If proper choice is made for the fluxes J_i and the forces X_i, the Onsager relations (5) are satisfied. The proper choice of forces and fluxes is obtained simply by expressing the entropy production σ in the form

$$\sigma = \sum_{i=1}^{n} J_i X_i = \sum_{i,j=1}^{n} L_{ij} X_i X_j \cdots\cdots (6)$$

J_i and X_i in the expression (6) are the conjugate fluxes and forces which satisfy the relations (4) and (5).

Among the many mehods of choosing the forces and fluxes which satisfy (6), we select the methods which made the coefficients $L_{ij}\,(i \neq j) = 0$, thus among the phenemenological coefficients there remain only the diagonal terms $L_{ii}\,(i = 1, 2, \cdots\cdots, n)$.

If we denote forces, fluxes and phenomenological coefficients in the diagonal form by X'_i, J'_i, and L'_i, the relations (4) and (6) would be transformed into the relations

$$J'_i = L'_i X'_i \cdots\cdots (7)$$

$$\sigma = \sum_{i=1}^{n} J'_i X'_i = \sum_{i=1}^{n} L'_i X'^2_i \cdots\cdots (8)$$

respectively.

If in any irreversible process, the forces X_i and the fluxes J_i which satisly (4), (5) and (6) are determined, we can deduce the forces X'_i and the fluxes J'_i which satisfy (7) and (8) as the following.

We put $X'_i = \sum_{k=1}^{n} \alpha_{ik} X_k$, and determine α_{ik}, L'_i by using the given conditions, then the forces X'_i and the fluxes J'_i which satisfy (7) and (8) can be obtained.

The number of equations of conditions, which is equal to the number of $L_{ij} = L_{ji}$ in the equality (6)=(8), is $\frac{n(n+1)}{2}$.

The number of unknown coefficients in the expression (8) is the sum of n (number of L'_i) and n^2 (number of α_{ik}), $n(n+1)$. Therefore, we can give some proper values to the coefficients of the number $n^2+n-\frac{n(n+1)}{2} = \frac{n(n+1)}{2}$, and determine the other coefficients of the number $\frac{n(n+1)}{2}$ by the equations of conditions.

As a simple method of determining the coefficients L'_i and α_{ik}, we put α_{ii} $(i=1,2,\cdots\cdots,n)$ equal to 1, and half the number of α_{ik} $(i\neq k)$ equal to 0 {the whole number of α_{ik} $(i\neq k)$ is $n(n-1)$}. Then L'_i $(i=1,2,\cdots\cdots,n)$ and the other half the number of α_{ik} are determined by the equations of conditions.

Of course, if we put different values to the coefficients of the number $\frac{n(n+1)}{2}$, the values of the other half coefficients woud be determined differently, thus we klnow that there are many methods of transformation of the forces and the fluxes into diagonal form.

We will treat two simple examples in the following.

Ex. 1. The transformation into diagonal form of two forces X_1, X_2 and two fluxes J_1, J_2 (the case of $n=2$).

In this case, there are relations

$$J_1 = L_{11} X_1 + L_{12} X_2 \quad \cdots\cdots (9)$$

$$J_2 = L_{21} X_1 + L_{22} X_2 \quad \cdots\cdots (10)$$

$$L_{12} = L_{21} \quad \cdots\cdots (11)$$

$$\sigma = J_1 X_1 + J_2 X_2 = L_{11} X_1^2 + 2 L_{12} X_1 X_2 + L_{22} X_2^2 \quad \cdots\cdots (12)$$

If we denote the forces, fluxes and phenomenological coefficients in a diagonal form by $(X'_1\ X'_2)$, $(J'_1\ J'_2)$, and $(L'_1\ L'_2)$, there are relations.

$$J'_1 = L'_1\ X'_1 \quad \cdots\cdots (13)$$

$$J'_2 = L'_2\ X'_2 \quad \cdots\cdots (14)$$

$$\sigma = J'_1\ X'_1 + J'_2\ X'_2 = L'_1\ X'^2_1 + L'_2\ X'^2_2 \quad \cdots\cdots (15)$$

To determine X'_1, X'_2, J'_1, J'_2, L'_1 and L'_2 we put

$$X'_1 = X'_1 + \alpha_{12} X_2 \qquad \left\{ \alpha_{11} = \alpha_{22} = 1 \right\} \quad \cdots\cdots (16)$$

$$X'_2 = X_2 \qquad \left\{ \alpha_{21} = 0 \right\} \quad \cdots\cdots (17)$$

in the expression (15), and equate it with (12), then we get the relation.

$$L_{11} X_1^2 + 2 L_{12} X_1 X_2 + L_{22} X_2^2 = L_1' (X_1 + \alpha_{12} X_2)^2 + L_2' X_2^2 \quad \cdots\cdots\cdots\cdots \quad (18)$$

comparing the coefficients, we get

$$L_1' = L_{11}, \quad \alpha_{12} = \frac{L_{12}}{L_{11}}, \quad L_2' = L_{22} - \frac{L_{12}^2}{L_{11}},$$

and

$$\left.\begin{aligned} X_1' &= X_1 + \frac{L_{12}}{L_{11}} X_2 \\ X_2' &= X_2 \\ J_1' &= J_1 \\ J_2' &= J_2 - \frac{L_{12}}{L_{11}} J_1 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (19)$$

These values satisfy (13), (14) and (15). Also we can transform in the following way. We put

$$X_1'' = X_1, \quad X_2'' = X_2 + \alpha_{21} X_1, \quad J_1'' = L_1'' X_1'', \quad J_2'' = L_2'' X_2'' \quad \cdots\cdots\cdots \quad (20)$$

and determine the values of X_1'', X_2'', J_1'', J_2'', L_1'' and L_2'' so as to satisfy the relation $\sigma = L_1'' X_1''^2 + L_2'' X_2''^2$, then we get the relations

$$L_2'' = L_{22}, \quad \alpha_{21} = \frac{L_{12}}{L_{22}}, \quad L_1'' = L_{11} - \frac{L_{12}^2}{L_{22}}$$

and

$$\left.\begin{aligned} &X_1'' = X_1, \quad X_2'' = X_2 + \frac{L_{12}}{L_{22}} X_1, \\ &J_1'' = \left(L_{11} - \frac{L_{12}^2}{L_{22}}\right) X_1 = J_1 - \frac{L_{12}}{L_{22}} J_2, \quad J_2'' = J_2 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (21)$$

Ex. 2. Transformation of three forces and three fluxes into a diagonal form ($n = 3$)

We denote the forces and the fluxes by X_1, X_2, X_3 and J_1, J_2, J_3, which satisfy the relations

$$\left.\begin{aligned} J_1 &= L_{11} X_1 + L_{12} X_2 + L_{13} X_3 \\ J_2 &= L_{21} X_1 + L_{22} X_2 + L_{23} X_3 \\ J_3 &= L_{31} X_1 + L_{32} X_2 + L_{33} X_3 \\ \sigma &= J_1 X_1 + J_2 X_2 + J_3 X_3 = L_{11} X_1^2 \\ &\quad + 2 L_{12} X_1 X_2 + 2 L_{13} X_1 X_3 + L_{22} X_2^2 + 2 L_{23} X_2 X_3 + L_{33} X_3^2 \end{aligned}\right\} \quad \cdots\cdots \quad (22)$$

To transform these into a diagonal form we assume the forces X_1', X_2', X_3' and the fluxes J_1', J_2', J_3' so as to satisfy the relations

$$\left.\begin{aligned} &J_1' = L_1' X_1', \quad J_2' = L_2' X_2', \quad J_3' = L_3' X_3' \\ &\sigma = J_1' X_1' + J_2' X_2' + J_3' X_3' = L_1' X_1'^2 + L_2' X_2'^2 + L_3' X_3'^2 \end{aligned}\right\} \cdots\cdots\cdots (23)$$

put $$X_1' = X_1, \quad X_2' = X_2 + \alpha_{12} X_1, \quad X_3' = X_3 + \alpha_{13} X_1 + \alpha_{23} X_2 \cdots\cdots\cdots (24)$$

$$\left(\begin{aligned} &\alpha_{11} = \alpha_{22} = \alpha_{33} = 1 \\ &\alpha_{21} = \alpha_{31} = \alpha_{32} = 0 \end{aligned}\right)$$

in the expression of σ, then we get

$$\begin{aligned} \sigma &= L_1' X_1^2 + L_2' (X_2 + \alpha_{12} X_1)^2 + L_3' (X_3 + \alpha_{13} X_1 + \alpha_{23} X_2)^2 \\ &= L_{11} X_1^2 + 2 L_{12} X_1 X_2 + 2 L_{13} X_1 X_3 + L_{22} X_2^2 + 2 L_{23} X_2 X_3 + L_{33} X_3^2 \end{aligned}$$

Comparing the coefficients, we get the relations

$$L_{11} = L_1' + L_2' \alpha_{12}^2 + L_3' \alpha_{13}^2, \quad L_{22} = L_2' + L_3' \alpha_{23}^2, \quad L_{33} = L_3'$$

$$L_{12} = L_2' \alpha_{12} + L_3' \alpha_{13} \alpha_{23}, \quad L_{13} = L_3' \alpha_{13}, \quad L_{23} = L_3' \alpha_{23}$$

From these relations, we can determine the values of L' and α as follows.

$$\left.\begin{aligned} &L_3' = L_{33}, \quad \alpha_{13} = \frac{L_{13}}{L_{33}}, \quad \alpha_{23} = \frac{L_{23}}{L_{33}} \\ &L_2' = L_{22} - \frac{L_{23}^2}{L_{33}} \\ &\alpha_{12} = \frac{1}{L_2'}\left(L_{12} - \frac{L_{13} L_{23}}{L_{33}}\right) = \frac{L_{12} L_{33} - L_{13} L_{23}}{L_{22} L_{33} - L_{23}^2} \\ &L_1' = L_{11} - \frac{(L_{12} L_{33} - L_{13} L_{23})^2}{L_{33} (L_{22} L_{33} - L_{23}^2)} - \frac{L_{13}^2}{L_{33}} \end{aligned}\right\} \cdots\cdots\cdots (25)$$

From these values we get

$$\left.\begin{aligned} &X_1' = X_1, \quad X_2' = X_2 + \frac{L_{12} L_{33} - L_{13} L_{23}}{L_{22} L_{33} - L_{23}^2} X_1 \\ &X_3' = X_3 + \frac{L_{13}}{L_{33}} X_1 + \frac{L_{23}}{L_{33}} X_2 \\ &J_1' = L_1' X_1' = \left(L_{11} - \frac{(L_{12} L_{33} - L_{13} L_{23})^2}{L_{33} (L_{22} L_{33} - L_{23}^2)} - \frac{L_{13}^2}{L_{33}}\right) X_1 \\ &J_2' = L_2' X_2' = \left(L_{22} - \frac{L_{23}^2}{L_{33}}\right)\left(X_2 + \frac{L_{12} L_{33} - L_{13} L_{23}}{L_{22} L_{33} - L_{23}^2} X_1\right) \end{aligned}\right\} \cdots\cdots\cdots (26)$$

$$J_3' = L_3' X_3' = L_{33}\left(X_3 + \frac{L_{13}}{L_{33}} X_1 + \frac{L_{23}}{L_{33}} X_2\right) = J_3$$

Of course, there are many other methods of transformation also in this case.

Thus we can always transform the general form

$$J_i = \sum_{j=1}^{n} L_{ij} X_j, \quad L_{ij} = L_{ji}$$

$$\sigma = \sum_{i=1}^{n} J_i X_i = \sum_{i,j=1}^{n} L_{ij} X_i X_j$$

into the diagonal form

$$J'_i = L'_i X'_i, \quad \sigma = \sum_{i=1}^{n} J'_i X'_i = \sum_{i=1}^{n} L'_i X'^2_i$$

by any proper linear transformation of forces.

5. Stationary States of Various Order

For simplicity, we denote the forces, the fluxes and the phenomenological coefficients which are expressed in a diagonal form, by X_i $(i=1,2,\cdots\cdots,n)$, J_i and L_i hereafter. Then there are the relations.

$$J_i = L_i X_i \quad (i=1,2,\cdots\cdots,n) \cdots\cdots (27)$$

$$\sigma = \sum_{i=1}^{n} J_i X_i = \sum_{i=1}^{n} L_i X_i^2 \cdots\cdots (28)$$

From (27) we know that J_i is determined by X_i only, and the change of J_i is produced by the change of X_i only. We may, therefore, study the variation of $J_i(X_i)$ by considering only the variation of $X_i(J_i)$. As J_i, which is produced by X_i, moderates X_i, X_i decreases gradually and becomes 0, if it is not fixed.

If X_i changes under the action of fixed applied forces X_{ia}, the difference of the fluxes from the fluxes of the stationary state, $J_i - J_{ia}$, moderates $X_i - X_{ia}$, and $|X_i - X_{ia}|$ decreases gradually and finally attain to stationary state in which the forces and fluxes are expressed by

$$X_i = X_{ia}, \quad J_i = L_i X_{ia} = J_{ia} \cdots\cdots (29)$$

respectively. Accordingly, if among n forces, $k(<n)$ forces X_i $(i=1,2,\cdots\cdots,k)$ are under the action of fixed applied forces X_{ia} $(i=1,2,\cdots\cdots,k)$ respectively, and the other forces X_j $(j=k+1, k+2,\cdots\cdots,n)$ are freely changeable, the moderation of forces by fluxes will occur and stationary state would be attained ultimately in which the forces and the fluxes

have the values,

$$X_i = X_{ia},\ J_i = J_{ia} = L_i X_{ia},\ X_j = 0,\ J_j = 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (30)$$

If $k=0,1,2,\cdots\cdots$, the stationary state is called the 0th, the first, the second, $\cdots\cdots$ order respectively.

In the stationary state of the 0th order, there is no force and no flux, so that it means thermostatic equilibrium state.

In the stationary state of the first, the second, $\cdots\cdots$ order, there exist one, two, $\cdots\cdots$ fixed forces and the conjugate constant fluxes.

As above mentioned, if we express the forces and the fluxes in a diagonal form, we can treat the stationary states easily. We can also compare easily the entropy production of the stationary state to that of the non-stationary state.

In the stationary state of the 0th order, all the fluxes are 0, and the entropy production is also 0, and therefore, when any perturbation is applied, the entropy production becomes positive, and afterwards decreases gradually to 0, value of the initial state, so that it can be said that the value of the entropy production is minimum in the stationary state of the 0th order. But in the stationary states of the 1st. 2nd, $\cdots\cdots$ orders, it can not be said that the value of the entropy production is minimum always. If the state of a system is stationary, in which the forces X_i $(i=1,2,\cdots\cdots,k)$ are fixed to the values X_{ia}, and the forces X_j $(j=k+1,k+2,\cdots\cdots,n)$ are 0, the perturbation which is given to any of the forces X_j, would necessarily increase the entropy production but the perturbation which is given to any of the forces X_i, would increase or decrease the entropy prodnction according as the strength of the force would increase or decrease, thus it can not be said that the entropy production is minimum in the stationary state always.

For example, when a fixed voltage is applied to a circuit, the electric current increases gradually from 0 to a fixed stationary value owing to electromagnetic induction, thus the entropy production increases from 0 to a stationary value.

If a magnetic perturbation is given to that stationary state, a new volage will be induced in the direction or in the opposite direction to the initial voltage, thus the entropy production will be increased or decreased according as the direction of the induced voltage, and therefore σ can not be said to be minimum in this case.

In these cases, the forces, the fluxes and the entropy production in the non-stationary state will generally tend to that of the stationary state. We will consider in the next paragraph the variation of forces, fluxes and entropy production with time in the non-stationary state.

6. Time Variation of Forces, Fluxes and Entropy Production in Non-stationary State

Generally, a flux moderates the conjugate force, and the effect of moderation is looked upon as almost proportional to the strength of flux in the scope of small flux.

For example, (a) If we connect by a fine resistance wire two insulatad conductors

whose potentials have a difference each other, the potential difference decreases proportionally to the strength of the flux (electric current). As the force is proportional to the potential difference, it can be said that the decrease of force per unit time is proportional to the flux.

(b) The heat flow between two bodies of different temperatures, reduces the difference of temperatures. As the forces is nearly proportional to the temperature difference, it can be said that the decrease of force is proportional to the flow of heat.

(c) When we connect two vessels which contain gases of different pressures with a narrow tube, the flow of gas reduces the pressure difference which is nearly proportional to the force. Therefore the decrease of force per unit time is nearly proportional to the flux (flow of gas).

Thus it can be said that the variation of a force per unit time caused by the conjugate flux, is proportional to the flux and has the opposite sign to the flux in the case of force freely changeable.

When the force varies under the fixed applied force, as the state becomes stationary at last, it can be said that

The variation of a force per unit time caused by the conjugate flux is proportional to the difference of the flux from that of the stationary state and has the opposite sign to the difference.

Thus, it is considered generally right that the relation

$$\frac{dX_i}{dt} = \alpha_i \, (J_{ia} - J_i) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (31)$$

will exist, where α_i is a proportional constant and J_{ia} is the flux of the stationary state.

If we denote the fixed force which produces the stationary flux by X_{ia}, there are the relations $J_{ia} = L_i X_{ia}$ and $J_i = L_i X_i$, and so we get the following relation by putting these in (31).

$$\frac{dX_i}{dt} = \alpha_i L_i (X_{ia} - X_i) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (32)$$

From this, we get

$$X_i - X_{ia} = (X_{io} - X_{ia}) \; e^{-\alpha_i L_i t} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (33)$$

where X_{io} is the initial value of X_i.

Multiply the both sides of (33) by L_i, then it is obtained for flux J_i the following relation.

$$J_i - J_{ia} = (J_{io} - J_{ia}) \; e^{-\alpha_i L_i t} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (34)$$

where J_{io} is the initial value of the flux. The entropy production is

$$\sigma = \sum_{i=1}^{n} L_i X_i^2 = \sum_{i=1}^{n} L_i \left\{ X_{ia} + (X_{io} - X_{ia}) e^{-\alpha_i L_i t} \right\}^2 \quad \cdots\cdots (35)$$

and the time variation of σ is

$$\frac{d\sigma}{dt} = 2 \sum_{i=1}^{n} \alpha_i L_i^2 (X_{ia} - X_{io}) e^{-\alpha_i L_i t} \left\{ X_{ia} - (X_{ia} - X_{io}) e^{-\alpha_i L_i t} \right\}$$

From these we know that if $X_{io} < X_{ia}$, σ decreases with time, and becomes minimum at the stationary state, but if $X_{io} > X_{ia}$, σ increases with time, and becomes maximum at the stationary state.

Thus the decrease or increase of σ follows the negative or positive signs of the value of $X_{ia} - X_{io}$.

After a stationary state is attained under some fixed forces, if some sudden changes of forces X_{ia} to another constant values X'_{ia} will occur, we know by (33) that the forces X_i will change according as the expression

$$X_i - X'_{ia} = \left(X_{ia} - X'_{ia} \right) e^{-\alpha_i L_i t} \quad \cdots\cdots (36)$$

The fluxes and the entropy production will change according as the following expressions respectively.

$$J_i - J'_{ia} = (J_{ia} - J'_{ia}) e^{-\alpha_i L_i t} \quad \cdots\cdots (37)$$

$$\sigma = \sum_{i=1}^{n} L_i \left\{ X'_{ia} + (X_{ia} - X'_{ia}) e^{-\alpha_i L_i t} \right\}^2 \quad \cdots\cdots (38)$$

where J'_{ia} are the stationary values of J_i conjugate to X'_{ia}.

Next if we multiply the both sides of (32) with L_i, we obtain the following expression

$$\frac{dJ_i}{dt} = \alpha_i L_i^2 (X_{ia} - X_i) \quad \cdots\cdots (39)$$

which is transformed into the expression

$$X_{ia} - \frac{1}{\alpha_i L_i^2} \frac{dJ_i}{dt} = \frac{J_i}{L_i} \quad \cdots\cdots (40)$$

From this expression we know that when a change in a flux is produced, the new force, $-\frac{1}{\alpha_i L_i^2} \frac{dJ_i}{dt}$, occurs in the direction of opposing the change of the flux.

Though the expressions (31) and (40) have different forms, and seem to have differ-

ent physical meanings, it can be said that they have the same physical meaning, and they both agree with the Le Chatelier-Braun's principle or Lenz's law.

7. Conclusion

According to de Groot, Prigogine and Onsager, if we denote the forces and the conjugate fluxes by X_i $(i=1,2,\cdots\cdots,n)$ and J_i, there are relations, $J_i = \sum_{j=1}^{n} L_{ij} X_j$, $L_{ij} = L_{ji}$, and the entropy production σ is expressed by $\sigma = \sum_{i,j=1}^{n} L_{ij} X_i X_j$.

By proper linear transformation of forces, we obtain diagonal form of expressions, in which the coefficients L_{ik} $(i \neq k) = 0$. Then we obtain the simple relations $J_i = L_i X_i$, $\sigma = \sum_{i=1}^{n} L_i X_i^2$. We treated the stationary states by this simple form, and made clear that the entropy production σ of the stationary states is not necessarily minimum.

We introduced a formula which expresses the relation between the flux and the variation of the conjugate force per unit time in a non-stationary state, and made clear the changes of the forces, the fluxes and the entropy production with time in the non-stationary state.

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THERMODYNAMICAL CONSIDERATIONS ON THE CELL DIVISION AND ON THE GROWTH OF ORGANISM

Shigeichi FUJITA

(Received Sept. 30 1954)

Abstract

The writer studied the cell division themodynamically and made its mechanism clear why the cell divides when it grows to a certain size. Further, he applied the same idea to the growth of organism, and obtained an expression for it. He compared this expression with that of Pütter-Bertalanffy, and could decide that the former is more suitable than the latter for living bodies practically.

1. Introduction

Vital phenomena are extremely complex, and there are many physiological functions which can not be explained even by the theories of the modern highly developed physics and chemistry. Some biologists had assumed the presence of a vital force, which has a mysterious super-physical action, so as to explain the vital phenomena. But the vital force would not give any comprehensive explanation about the phenomena. Vital force takes only the place of the vital phenomena, and is still beyond comprehension after all. We must analyse the mysterious vital force and make it clear by the power of the modern science. Although, in vital phenomena, many physical and chemical changes are combined in very complex manner, each phenomenon is to be explained by the laws of physics and chemistry. In this point of view, we study the cell division and the growth of organism physically.

In the cell division, which is one of the fundamental vital phenomena, the induction of the nuclear division has been studied by many authors and the theory of mitogenetic rays (Gurwitsch) and the theory of cell division hormone (Haberlandt) have been presented. About the mechanism of nuclear division, the fibrillar hypothesis, the stream hypothesis and the electric hypothesis, and etc. have been presented and the movement of the chromosome has been explained physically from the point of each hypothesis. But about the mechanism of the cytosome division, little attention has been payed until now. The writer wants to study thermodynamically the conditions of the cytosome division and wishes to promote the comprehension about the vital phenomena.

2. Free Energy of a Cell

As a cell is a living body, we must not consider it like a simple homogeneous liquid drop. The structure of a cell is very much complex, functional and exquisite

compared with a simple liquid drop. When a cell grows and increases its energy, the energy increase must not be considered like that of a simple water drop when it becomes larger.

When a cell grows, it is considered that the intracellular energy density would increase, say in other words, there would exist long range forces among the corpuscles which construct the cell, and the range of action of a corpuscle would become larger and larger as the cell grows, thus the energy per one corpuscle increases with the growth of the cell.

We consider the free energy of a cell as consisted of two parts, the surface energy and the intracellular energy. The surface energy is looked upon as proportional to the surface area, if the surface has the same property everywhere. The intracellular energy can not be looked upon as proportional to the volume like the case of a simple water drop. The nucleus and the cytoplasm are not homogeneous, and are to be considered that the degree of complexity of the structure increases with the growth of the cell, thus, when the volume of the cell increases, the intracellular energy increases much more than the case as if it increases proportionally to the volume. Therefore, if we denote the free energy, the surface area and the volume of a cell by G, o and v respectively, G will be expressed nearly by the following formula

$$G = a'o + bv^n, \quad n > 1 \qquad \cdots\cdots (1)$$

or

$$G = a'o + b'v + cv^2 + dv^3 + \cdots\cdots (2)$$

where $a', b, b', c, d, \cdots\cdots$ are the constants. If the cell grows proportionally, we can put

$$G = av^{2/3} + bv^n, \quad n > 1 \qquad \cdots\cdots (3)$$

or

$$G = av^{2/3} + b'v + cv^2 + dv^3 + \cdots\cdots (4)$$

where a is a constant, different from a'. For simplicity, we shall consider only the case of the proportional growth. (3) or (4) is expressed graphically as in the Fig. 1.

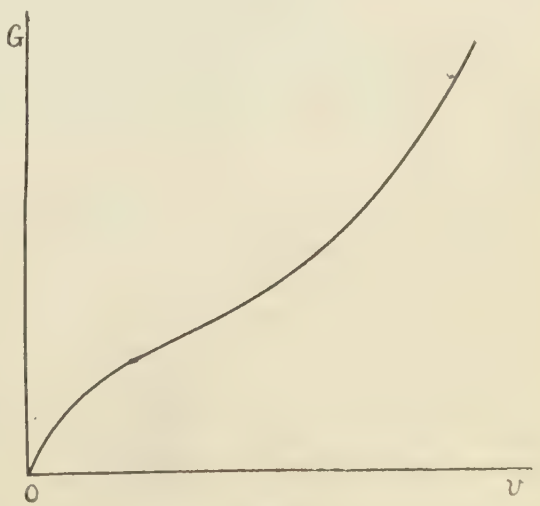

Fig. 1

3. Thermodynamical Consideration on the Cell Division

If the free energy of a cell is given by the expression (3) or (4), we would be able to conclude that the cell division would occur inevitably when the cell grows to a certain size.

The free energy of an isolated

system decreases with the irreversible processes in it, and the state of a system of small free energy is more stable than that of large free energy. If, therefore, the free energy of a cell of a certain large size is greater than the sum of the free energies of two small cells which are produced by one division of the large cell, the state of two small cells are more stable than that of one large cell, and the division occurs at that large size spontaneously.

If we denote the volume of a small cell, which is produced by one division of a cell whose free energy is expressed by (3), by v_1, and the sum of the free energies of the two small cells by G_1, we obtain the next expression, by using the relation $v_1 \fallingdotseq \frac{v}{2}$

$$G_1 = 2\left(av_1^{2/3} + bv_1^n\right) = av^{2/3}2^{1/3} + bv^n 2^{1-n} \quad \cdots\cdots (5)$$

If we imagine a cell whose surface has the same property everywhere, and whose interior has the uniform and isotropic property, like a simple homogeneous liquid drop, we must take $n = 1$ for that cell and we know that the relation $G_1 > G$ always applies, and the division would not occur spontaneously. Of course, when n is smaller than 1, $G_1 > G$ always holds.

When n is greater than 1, in the expressions of G and G_1, the first term is predominate while v is small and the relation $G_1 > G$ holds, but when v becomes large, the second term becomes predominate and the relation $G_1 < G$ holds and therefore the division of the cell occurs. The size of the volume V of a cell when the division begins to occur, is obtained as the value of v which satisfies the relation $G_1 = G$, as in the following way.

$aV^{2/3} + bV^n = aV^{2/3}2^{1/3} + bV^n 2^{1-n}$

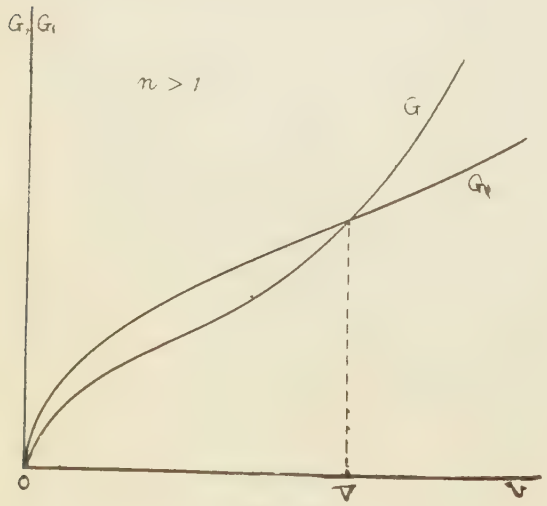

Fig. 2

$$V = \left\{\frac{a}{b}\cdot\frac{2^{1/3}-1}{1-2^{1-n}}\right\}^{\frac{3}{2n-2}} \quad \cdots\cdots (5)$$

The relation between G, G_1 and v is also expressed graphically in Fig. 2 The value of v when the two curves intersect each other denote the value of V.

When v is smaller than V, the curve of G_1 is above that of G, thus the state of one cell is more stable than that of the divided two cells. When v is larger than V, the curve of G_1 is below that of G, thus the state of one cell is unstable and division occurs.

It is considered as a special case that in a stable cell some biochemical changes occur and some energy becomes free and the cell sometimes becomes unstable and divides. As an example of this case it is stated the fact that the fertilized ovum continues to divide many times without absorption of any nourishment.

When $n \leqq 1$ is satisfied, G-curve and G_1-curve diverge each other and do not intersect as seen in Fig. 3.

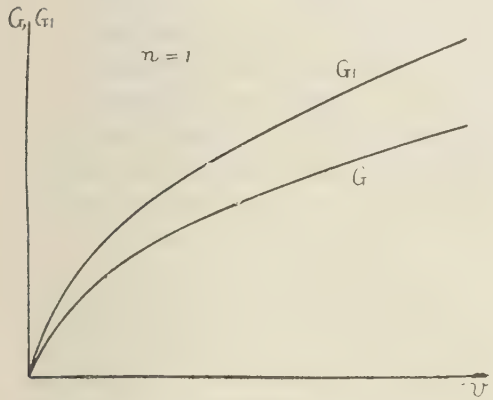

Fig. 3 (a)

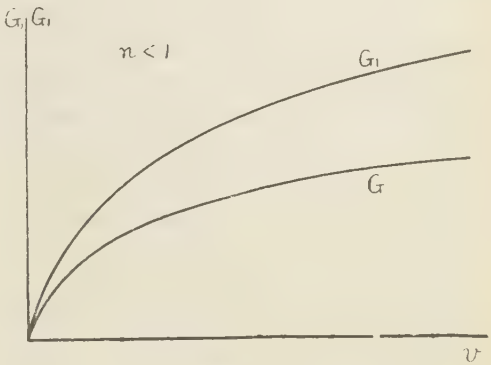

Fig. 3 (b)

The fact that the cells take forms other than sphere would have, I suppose, an intimate relation to the intracellular complex structure and the energy of the type av^n $(n > 1)$. For example, when a group of corpuscles in any cell grows to a certain size and becomes unstable and divides in many groups, then the form of the cell would be deformed so as to satisfy the minimum condition of free energy of the cell.

If the free energy of a cell is composed of two terms, of which one is proportional to the surface area, and the other is proportional to the volume of the cell, sphere is the stable form and the other forms are less stable or unstable as in the case of a homogeneous liquid drop.

But we must notice the fact that a homogeneous liquid drop may divide under the influence of any external action. In the experiments of Mc Clendon and Spek, the energy of the liquid drop was increased by the application of the special external action, and the strength of the surface tension was changed at the particular part of the surface, thus the division of drop was brought about. In the case when large water drops divide in the current of air, the water drops receive some external action and acquire some sufficient energy for division. When a water drop does a self-vibration by any external action, the vibrational energy will be increased greatly as the size of the drop increases. The vibrational energy will supply the energy of division, so that it is considered that as the size of the drop becomes large, the chance of division will be increased. In short, it is possible that even an uniform water drop which has uniform surface energy and homogeneous internal energy,

would divide if any external action would exert on it.

Similarly, it is possible that a cell, without absorbing any nourishment, would divide if it is exerted by some special external action, if it is placed in a special boundary condition, and if any special biochemical process occurs and the free energy increases suddenly in the cell.

4. Thermodynamical Consideration on the Growth of Organism

Though the expression (3) or (4) has been given for a cell, it is considered right that (3) or (4) will be applied to any living body, if we give any different proper values to the constants $a, b, \cdots\cdots$. We consider the free energy of a living body composed of two parts, the surface energy and the internal energy, as in the case of a cell. The surface energy may be proportional to the surface area, but the internal energy would not be proportional to the volume of the body, as in the case of a cell. The degee of complexity, of functionality, or of regularity, would be promoted and the energy density would increase with the growth of the living body.

Living body absorbs nourishment from outside and performs assimilation. It is considered that the quantity of the absorption of nourishment or the assimilation is nearly proportional to the surface area of the body. In the plant, the absorption of nourishment from the roots and the carbon-assimilation performed in the leaves by the aid of sun light seem to be nearly proportional to the area of these parts. In some animals of low class, the quantity of absorbed nourishment may be nearly proportional to the surface area as in the cae of a cell. In the ordinary animals, the quantity of absorbed nourishment will be nearly proportional to the inside area of the intestine. In the proportional growth, the surface area of the animal will be proportional to the inside area of the intestine. Thus the quantity of absorption and assimilation is assumed to be nearly proportional to the surface area.

On the other hand, the dissimilation or the decomposition of substances by the vital actions is carried out in all the organs of the body, and the vital actions, consisting in the absorption and digestion of nourishment, assimilation and dissimilation, secretion and excretion, movement and transportation, regulation and nervous action, etc., are performed by the free energy contained in the body. It is consibered, therefore, that the decomposition of substances by the vital actions is nearly proportional to the free energy of the living body.

If the free energy of a living body is given by the expression similar to (3), it is considered that the quantity of substance, decomposed and excreted, is nearly proportional to $v^{n'}$.

Accordingly, we would be able to express the rate of growth of a living body in the following formula

$$\frac{dv}{dt} = \alpha v^{2/3} - \beta v^{n'}, \quad (n' > 1) \qquad \cdots\cdots (7)$$

where t is the time, α and β are some constants. It is considered that the value of α may vary with v as though $\alpha=\alpha_0-\gamma v^m$ (α_0, γ, m are some constants) in practice, but it is easily seen that by putting the relation into (7), the form of (7) would not be changed fundamentally.

By these reasons we want to propose the formula (7) as a new law of growth. The well-known Pütter-Bertalanffy's law of growth can be expressed in the form of the following formula.

$$\frac{dv}{dt} = \alpha v^{2/3} - \beta v \quad \cdots\cdots (8)$$

Comparing this with (7) we know that when we put $n' = 1$ in (7), we get (8), so that in (8) the dissimilation is assumed to be proportional to the volume of the body.

5. Comparison of the new Law of Growth with that of Pütter-Bertalanffy

As has been stated, in the case of $n'=1$, it is assumed that the interior of the living body is homogeneous, and if the body has no energy other than the uniform surface energy and the homogeneous internal energy, the body must take spherical form, and if we take a cell as the living body, the cell does not divide even when it grows to any large size unless it is acted by some special action from surrounding or some special biochemical process occurs in it. The growth given by (8) is to be applied to the living bodies thus assumed.

For living cells which divide at some growth and for living bodies whose interior construction is complex and exquisite, which can take any forms other than sphere, (7) is more reasonable than (8).

We shall further compare and study the rate of growth $\frac{dv}{dt} = \dot{v}$ given by (7) with that given by (8). The volume at the end of growth v_0, and the volume at the maximum rate of growth v_1, are obtained as the volumes which satisfy $\dot{v}=0$ and $\frac{d\dot{v}}{dv} = 0$ respectively. Accordingly, if the growth obeys (7), they are obtained as following

$$\alpha v_0^{2/3} - \beta v_0^{n'} = 0, \quad v_0 = \left(\frac{\alpha}{\beta}\right)^{\frac{3}{3n'-2}} \quad \cdots\cdots (9)$$

$$\frac{2}{3}\alpha v_1^{-1/3} - n'\beta v_1^{n'-1} = 0, \quad v_1 = \left(\frac{2}{3}\frac{\alpha}{n'\beta}\right)^{\frac{3}{3n'-2}} = \left(\frac{2}{3n'}\right)^{\frac{3}{3n'-2}} v_0 \quad \cdots\cdots (10)$$

If the growth obeys (8), they are obtained as following

$$\alpha v_0^{2/3} - \beta v_0 = 0, \quad v_0 = \left(\frac{\alpha}{\beta}\right)^3 \quad \cdots\cdots (11)$$

$$\frac{2}{3}\alpha v_1^{-1/3}-\beta=0,\quad v_1=\left(\frac{2\alpha}{3\beta}\right)^3=\frac{8}{27}v_0 \qquad (12)$$

The raito $k=\frac{v_1}{v_0}=\left(\frac{3}{2n'}\right)^{\frac{3}{3n'-2}}$ increases with the increase of n' as seen in the following table.

n'	1	2	3	4	5	6
k	$\frac{8}{27}(=0.296)$	0.439	0.525	0.584	0.628	0.662

If the growth obeys (8), k is $\frac{8}{27}$, i.e., the rate of growth is maximum when the volume of the body is $\frac{8}{27}$ of that of the end of growth.

In practice, the ratio of the volume of the body at the time of maximum rate of growth to that at the time of the end of growth is discovered to be greater than $\frac{8}{27}$ in many cases. This fact shows in our opinion, that n' is greater than 1 in many living bodies. Further, if the proportional growth of a single period of a living body is observed exactly, from the value of k, the ratio of the volume at the time of maximum rate of growth to the volume at the time of the end of growth, n' is obtained for that body. As n' has some intimate relation to the degree of the complexity, functionality and regularity, it must be able to conjecture the degree of evolution. If there are living bodies whose growth obey (8), they are supposed to be the bodies of lowest class, or they are looked upon as bodies hardly to be said living.

In the expression of growth of Robertson which was applied to the growth of many living bodies and gave important results, the ratio of the volume at the time of the maximum rate of growth to the volume at the time of end of growth is 1/2. Many observed examples of the growth are the cases of complex growth, and can not be looked upon as any actual proof to decide which of (7) or (8) is right for the living bodies. But it is almost conjectured from many observed results that the value of k is generally greater than $\frac{8}{27}$, hence n' is greater than 1.

In the following, several examples are stated.

(i) H. S. Reed[8] measured the growth of rat in each period, and obtained the formula which express esthe relation between the body weight $x\,(g)$ and the days t after its birth. Among the formula only those of the growth of the first period are given in the following.

In the first period of male rat

$$\log\frac{x}{228-x}=0.0187\,(t-73) \qquad (13)$$

In the first period of female rat

$$\log\frac{x}{170-x}=0.0211\,(t-61) \qquad (13')$$

From the expressions we know the body weight x_0 at the time of the end of growth is 228g in the case of male rat, 170g in the case of female rat. The body weight x_1 at the time of maximum rate of growth is $\frac{228}{2}$ g = 114g in the case of male rat, and $\frac{170}{2}$ g = 85g in the case of female rat respectively, hence the ratio $\frac{x_1}{x_0}$ is 1/2, greater than $\frac{8}{27}$.

(ii) T. B. Robertson[4] measured the growth of oats, a part of the results is given in the following table

days	0	4	8	16	21	25	29	35	39	42	47	51	57	64
weight (g)	0.94	1.78	3.58	14.51	34.70	52.20	141.2	185.7	243.3	314.3	304.0	254.6	229.9	224.3
increase of weight per day		0.21	0.45	1.36	4.04	4.37	22.3	7.4	14.4	25.0	−2.1	−12.3		

Although it is difficult to determine the value of v_1 and v_0 from the table, it is conjectured almost nearly that there are more than two periods, in the first period it might be taken v_0 ∴ 180g, v_1 ∴ 100g and $\frac{v_1}{v_0}$ ∴ $\frac{5}{9}$ $\left(> \frac{8}{27}\right)$, and the second period begins before the end of the first period.

(iii) According to British Association Report*, the growth of man is as follows.

years	0.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5	11.5	12.5	13.5
weight (kg)	3.4	18.6	22.7	24.6	25.9	27.0	28.3	30.4	32.3	35.0	37.9
increase of weight per year		3.8	4.1	1.9	1.3	1.1	1.3	2.1	1.9	2.7	2.9
years	13.5	14.5	15.5	16.5	17.5	18.5	19.5	20.5	21.5	22.5	
weight (kg)	37.9	41.7	47.1	53.8	59.6	62.5	64.4	66.5	67.1	67.2	
increase of weight per year		3.8	5.4	6.7	5.8	2.9	1.9	2.1	0.6	0.1	

From this table it is conjectured almost nearly that there are more than 2 periods, the first period ends at 7, 8 years old, the second period begins before the end of the first period, and in the second period it might be taken as v ∴ 67kg, v_1 ∴ 50kg and $\frac{v_1}{v_0}$ ∴ 0.75 $\left(> \frac{8}{27}\right)$. But as the growth of man is very complex, it may be not right to use this table for determining n'.

From the examples mentioned above, we can not deduce any determinate couclusion. But it is considered almost surely that in many living bodies the relations, $\frac{v_1}{v} > \frac{8}{27}$ and $n' > 1$, are satisfied, thus for the living bodies the expression (7) is more suitable than the expression (8).

* cited from T. B. Robertson

6. Conclusion

Although, in homogeneous liquid, there is a tendency of union of small nearly existing drops to a large drop, in living cells, when they grow to some size, they divide usually. This fact reminds us that the living cells have fundamentally different aspect on their construction from homogeneous liquid drops.

The writer considered the difference from the standpoint of free energy and attained to an idea. By this idea the writer could conclude that the living cells would divide naturally when they grow to some sizes.

Further, as this idea has an intimate relation to the growth of organism, on the same foundation the writer obtained an expression for the growth of organism and compared it with the notable law of growth of Pütter-Bertalanffy, and could conclude almost nearly that the former is more suitable than the latter for living bodies.

In conclusion, the writer wishes to express sincere gratitude to Dr. G. Goto, Department of Biology, Kumamoto University, for his kind and copious instruction and advice about the biological phenomena, and for his kind lending many precious references and data.

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THE EFFECT OF THE MAGNITUDE OF THE DEMAGNETIZING ALTERNATE-FIELD ON THE LONGITUDINAL MAGNETOSTRICCION IN WEAK MAGNETIC FIELD.

Sigeo MATSUMAE

(Received Sept. 30 1954)

(I) Introuduction.

It is found in nickel and perminvar-alloy that the longitudinal magnetostriction in the weak magnetic field, in which the magnetic permeability is constant, is greatly affected by the condition of the internal stress in the specimen and by the method of demagnetization, demagnetizing the specimen by heating or by alternate-field, and also affected by the magnitude of the maximum amplitude of demagnetizing alternate-field.[1][2][3] On further, we study the effect of the magnitude of the demagnetizing alternate-field on the longitudinal magnetostriction in weak magnetic field. The experimental results are as follows.

(II) Experimeantal result and Discussion.

Experimental apparatus and method of measurement are the same as described in previous papers.

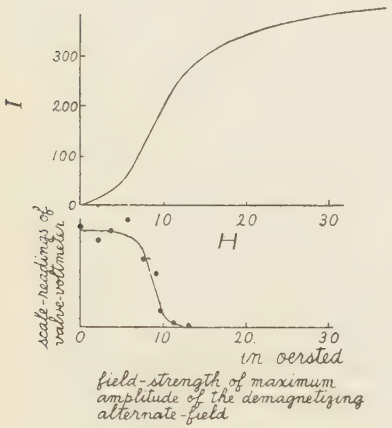
As the induced voltages in the secondary coil are proportional to the magnitudes of the longitudinal magnetostrction but the proportional constants are not same, whenever the same specimen is demagnetized by the alternate-field of different magnitude, the magnitudes of the longitudinal magnetostrictinon are not accuratly denoted by the magnitudes of the induced voltages in the secondary coil. But, the change of the proportionl constants being comparatively small, the rough tendency of the magnitude of longitudinal magnetostriction may be found in the tendency of the magnitude of the induced voltage in the secondary coil. As in previous papers, the induced voltage in the secondary coil is amplified by the audio-frepuency amplifier and then measured by the valve-voltmeter. Consequently, the induced voltages are denoted by the scale-readings of the valve-voltmeter in the figure.

The specimen is a round bar of the electrolytic nickel, containing about 0.3% manganese, 58cm in length and 6cm in diamter.

The specimen is cold-worked and then annealed at 800 C in about 10 minutes, finally being cooled to the room-temperature with the electric-furnace by cutting the heating-current off, as the magnitude of the longitudinal magnetostriction in weak magnetic field is, on an average, large in nickel in the case of such a heat-treatment.

Experimental results are shown in Fig. 1. In the lower part of the figure, the relation between the scale-reading of the valve-voltmeter when a small constant exciting

magnetic field is applied and the magnitude of the maximum amplitude of the demagnetizing alternte-field of 50 cycles per second is shown. Abscissa indicates the magnitude of the maximum amplitude of demagnetizing alternate-field and its origin indicates the alternate-field of 0 oersted, where the specimen is in the condition demagnetized by heating.



As shown in the lower part of the figure, it becomes clear that the longitudinal magnetostriction in very weak field, in which the magnetic permeability is constant, has such a tendency that it changes somewhat for demagnetizing by the alternate-field having a maximnm amplitude of from 0 from to about 7 oersted but begins to diminish rapidly just above this field-strength and, after the abrupt fall, diminishes graduall gradually with the increase of the field-strength.

In the upper part of the figure, I-H curve of the specimen is shown. The intensity of the magnetization is ballistically measured.

As shown in the figure, the abrupt fall of the magnitude of the longitudinal magnetostiction seems to occur at the part in which the irreversible boundary displacements take place, in the case of such a heat-treatment.

(III) Summary.

As shows in the previous, papers, the longitudinal magnetostriction in very weak magnetic field is greatly affected by the magnitude of the maximum amplitude of the demagnetizing alternate-field.

The present work was supported by the grant for Scientific Researches from Education Ministry.

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(I) S. Matsumae: Sci. Rep. of TOHOKU-Univsity. XXX I, IV
(II) S. Matsumae: KUMAMOTO-Journal of Science. Vol. I, No. I
(III) S. Matsumae: KUMAMOTO-Journal of Science. Vol. I, No. 4

THE INVESTIGATION OF "SARA-ISI" BY THE X-RAY (Report 2)

Tadashi OKAHATA

(Received September 30, 1954)

1. Introduction

About chemical constituents of the coagurated part of Sara-isi and the volcanic ashes, Dr. M. Namba has expressed the results of the chemical anaylysis in his work. [1]. Of cource there exists much silicic acid Sio_2 compared with other constituents in them. According to these chemical constituents, as imaginable crystalline elements in the coagurated part of Sara-isi and in the volcanic ashes, it can be thought that quartz, felspar, black mica, magnetite, pyroxene, olivine and hornblende are contained in them.

Debye-Scherrer's method was applied to the coagurated part of Sara-isi and to the volcanic ashes, and the weak Debye-Scherrer rings, which correspond to felspar only, was found in the radiograph.

felsper: $M^{I}AlSi_3O_8+M^{II}Al_2Si_2O_8$ where M^{I}: K, Na. M^{II}: Ca, Ba (rare).

Main crystalline element in the coagurated part of Sara-isi is felsper, and even if other crystalline elements as stated above are contained, their quantity are very small. Silicic acid exists mostly in the form of gel, not of quartz.

2. Specimen and Experimental Procedure

The coagurated part was cut from Sara-isi, and was crushed in the iron mortar and powered in the porcelaineous mortar, and besides, in the agate one. Especially in the last one, it was powered for about an hour.

A small quantity of this fine powder, thus obtained, was held between two sheets of cellophane on a deckglass and rolled to a thin plate about 0.2 mm in thickness by lightly rubbing on the upper sheet with a glass pole. It was very easy to separate this plate specimen from the sheet of cellophane. This thin specimen was attached directly to the slit with grease. Specimen of Sio. gel, quartz, felspar, mica, magnetite, pyroxene, olivine, hornblende, volcanic ashes and mud was prepared just the same as stated above. To all these specimens Debye-Scherrer's

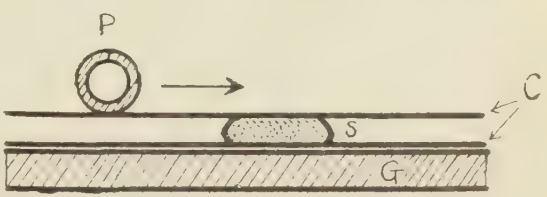


Fig. 1. P: glass pole
C: cellophane
S: fine powder
G: deck-glass

method (with plane photographic film) was applied and radiographs thus obtained were examined.

X-ray (50 KV, 15 MA) from Cu-target was filtered by Ni, 1/100 mm in thickness, and was used. The distance between the specimen and the photographic film was held about 40 mm, and photographic film was always set perpendicular to the incident x-ray beam.

3. Experimental Results

By the experimental procedure as stated above, many radiogaphs were obtained with all the specimens, and the radiographs which were picked out of them are reproduced in Figs. 2, 3, 4, 5, 6, 7 Plate 1.

Fig. 2 is a radiograph with a specimen which was obtained from the coagurated part of Sara-isi, Fig. 3 is with the volcanic ashes which was gathered in the neighbourhood of the creater of the volcano Aso, and Fig. 4 is with the mud which was gathered in Kumamoto on 26th. June in 1593. [2].

Rings in these figures coinside with each other, moreover, these rings correspond to the felspar. (Compare. Figs. 2, 3, 4 with 5). These aspects are reproduced in Figs. 8. 9. 10. 11, Plate II.

On the other hand, the radiograph with Sio_2 gel which was taken out chemically from the coagurated part of the Sara-isi is shown in Fig. 7. In Fig. 2. the central part is so hazy. This phenomena originate in SiO_2 gel which is contained in the coagurated part of Sara-isi.

Thus as stated in the introduction, the main crystalline element in the coagurated part of Sara-isi is felspar, and even if other crystalline elements are cotained, their quantity must be very small. And in this coagurated part, silicic acid are contained mostly in the form of gel, and not of quartz. It seems that, this arguments are applicable to the volcanic ashes and the mud.

The writer wishes to express his thanks to Dr, M. Namba who gave kind advices to him.

Reference

1. M. Namba; Mem. Col. Sci. Kyoto Imp. Uni. Vol. 19, No. 3 (1936)
2. T. Okahata; J. Sci. Kumamoto Uni. Vol. 1, No. 4 (1954)

Plate I

Fig. 2
rim of Sara-isi

Fig. 3
ashes

Fig. 4
mud

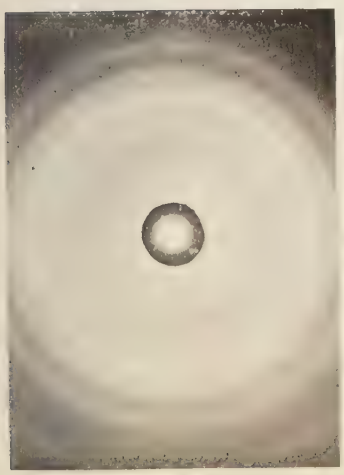


Fig. 5
felspar

Fig. 6
quartz

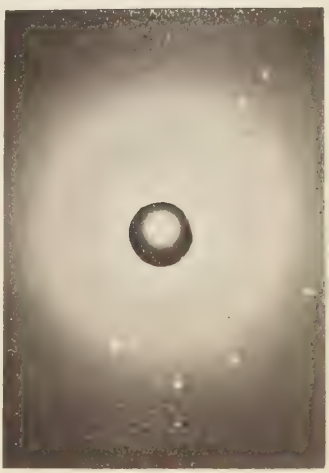


Fig. 7
S_iO gel

Plate II

Fig. 8

D B

Fig. 9

B C

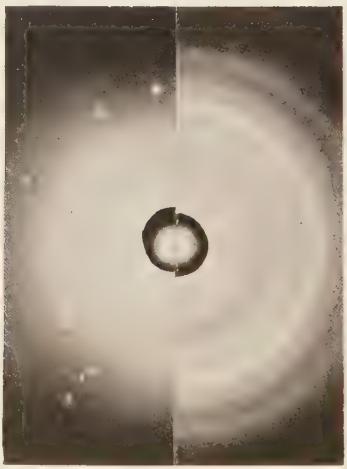
Fig. 10

B A

Fig. 11

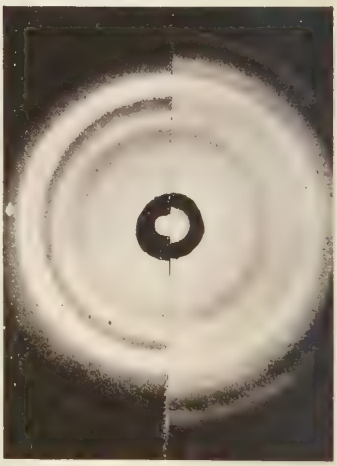
B A D C

Fig. 12



D SiO_2 gel.

Fig. 13



quartz. D

A: coagurated part of Sara-isi
B: volcanic ashes
C: mud
D: qelspar

ON THE BALLOONING PROBLEM (II)

Akira ONUKI Mituo ITÔ

4. The Experiment

The photographs of moving threads were taken by two cameras which were set on the floor in two directions perpendicular each other, so that they can catch the shape of thread in one instant by a flash of light. The curves obtained in this way were reduced and corrected in taking account of the small difference of focal lengths of the two cameras and the size contraction arising from the obliquity of the line of sight. Some of the interesting and important results are tabulated in the following tables 4, 5, 6 with the paramenters η and τ which were calculated from the eq. (1. 15)

Table 4

No.	1		2		3		4		5		6	
ω	3000		4000		3300		4200		2800		3500	
Z	ρ	θ	ρ	θ	ρ	θ	ρ	θ	ρ	θ	ρ	θ
0.1	0.138	0.°0	0.188	0.°0	0.149	0.°0	0.195	0.°0	0.131	0.°0	0.142	0.°0
0.2	0.289	2.7	0.370	0.1	0.291	1.7	0.360	0.0	0.274	0.0	0.263	3.4
0.3	0.441	3.6	0.519	0.9	0.436	1.7	0.519	0.7	0.395	0.1	0.392	6.4
0.4	0.567	4.1	0.669	1.4	0.542	1.7	0.676	1.1	0.502	0.2	0.511	6.6
0.5	0.706	5.5	0.808	2.3	0.668	1.7	0.807	2.9	0.617	0.3	0.625	6.9
0.6	0.788	5.7	0.902	3.4	0.780	2.5	0.896	5.2	0.724	1.3	0.727	8.3
0.7	0.889	6.8	0.974	5.4	0.874	3.4	0.972	8.1	0.820	2.0	0.817	9.3
0.8	0.926	8.8	1.003	7.6	0.936	5.5	1.013	10.9	0.898	3.4	0.901	10.7
0.9	0.980	11.1	1.013	11.6	0.980	8.2	1.013	15.4	0.960	5.1	0.962	11.9
1.0	1.000	13.4	1.000	15.2	1.000	11.7	1.000	20.3	1.000	8.1	1.000	14.2

Experimental Values of ρ and θ
ω = rotation per minute

Table 5

No.	7		8		9		10	
ω	2150		2000		2420		2080	
Z	ρ	θ	ρ	θ	ρ	θ	ρ	θ
0.1	0.234	0.°0	0.446	0.°0	0.464	0.°0	0.425	0.°0
0.2	0.460	0.5	0.779	0.6	0.573	9.2	0.708	0.0
0.3	0.663	1.0	0.937	2.1	0.388	44.8	0.837	5.9
0.4	0.837	2.9	0.865	9.0	0.507	138.2	0.801	16.1
0.5	0.976	4.4	0.601	24.8	0.802	172.7	0.569	39.1
0.6	1.074	6.5	0.369	81.7	0.705	201.6	0.449	96.3
0.7	1.125	7.6	0.572	137.6	0.568	266.6	0.675	145.3

0.8	1.121	10,6	0.881	158.5	0.795	333.5	0.950	170.2
0.9	1.092	13.6	1.026	171.7	1.052	380.7	1.052	188.7
1.0	1.000	19.5	1.000	187.5	1.000	426.2	1.000	208.1

ω = rotation per minute
Experimental Values of ρ and θ

Table 6

No	a	c	T_0 (g)	ω r.p.m	η	γ
1	55	6.05	15.0	$3\cdot10^3$	0.64525	1.5851
2	〃	〃	〃	$4\cdot10^3$	0.86034	2.8180
3	〃	〃	〃	$3.3\cdot10^3$	0.70977	1.9180
4	〃	〃	〃	$4.2\cdot10^3$	0.90336	3.1067
5	〃	〃	〃	$2.8\cdot10^3$	0.60224	1.3808
6	〃	〃	〃	$3.5\cdot10^3$	0.75280	2.1575
7	36.4	4.00	1.518	$2.15\cdot10^3$	0.96207	2.330
8	〃	〃	0.550	$2.00\cdot10^3$	1.48676	5.564
9	〃	〃	0.325	$2.42\cdot10^3$	2.34030	13.785
10	〃	〃	0.325	$2.08\cdot10^3$	2.01154	10.184

But it must be noted that the condition of these experiments are slightly different from the actual state, especially in one point, that in these experiments the both ends of thread are fixed and consequently it does not need to consider the moving effect of thread along itself. This situation has the advantage that one can measure the tension at one end of thread easily by hanging down the mass of known weight.

Mathematically, such modification of the measurements implies that it should be added to the orignal fundamental equations of section 1 one more additional equation which expresses the constancy of the total length of the thread between two ends. It is well known that the role of this last equation is to determine the tension of thread (e.g. T) absolutely. Accordingly we can expect that these modification does not cause any serious variation of ballooning figures if we compare them with the results of numerical integration in which the Corioli's term is entirely neglected. The angel θ given in the tables above is the position angle of the point of the thread from the X-Z plane, as is already stated. Increments of the values of θ from the point $Z=0$ to $Z=1$ are roughly 10° in the table 4 except the measurement No. 4, in that case it attains to about 20°. But in the table 5, one sees these are far more larger than former case. The reason of this difference comes from the smallness of T_0 (as is shown in the table 6), as one sees easily from the equations (1.7) (1.13) (1.15). In fact, the values of T_0 in No. 8 No. 9 & No. 10 are so small that the values of $\gamma, \tilde{\gamma}$ and $d\theta/dZ$ become fairly large. In such cases the comparison of the experiment and calculation should be done very carefully, and more complicated laborious works are required.

5. The comparison of the theory and experiments.

Since in the Table 4, & 5, no "licking" arises for the experiments No. 1-7, and besides, θ's change along whole length become no more than 20 degrees, it is worth to try to fit the solution by the elliptic function for these cases, choosing some suitable α_1 and γ. The numerical results reckoned in this way are proved to be satisfactory and are given below.

No. 1 $\rho = 1.0436\ Sn\ (1.2872\ Z,\ 0.00541)$
No. 2 $\rho = 1.0122\ Sn\ (1.7268\ Z,\ 0.00908)$
No. 3 $\rho = 1.0104\ Sn\ (1.4330\ Z,\ 0.00626)$
No. 4 $\rho = 1.0344\ Sn\ (1.8360\ Z,\ 0.01068)$
No. 5 $\rho = 1.0683\ Sn\ (1.2136\ Z,\ 0.00503)$
No. 6 $\rho = 1.0016\ Sn\ (1.5216\ Z,\ 0.00693)$

Table 7

No	1		2		3		4		5		6	
ω	3000		4000		3300		4200		2800		3500	
α_0	8°27′		11°02′		9°03′		11°48′		8°07′		9°31′	
Z	ρ	T_1	ρ	T_1	ρ	T_1	ρ	T_1	ρ	T_1	ρ	T_1
0.1	0.133	0.9998	0.173	0.9995	0.144	0.9997	0.189	0.9993	0.129	0.9999	0.152	0.9997
0.2	0.265	0.9993	0.342	0.9979	0.286	0.9990	0.371	0.9973	0.257	0.9994	0.300	0.9988
0.3	0.393	0.9984	0.501	0.9955	0.421	0.9978	0.541	0.9942	0.380	0.9987	0.441	0.9973
0.4	0.514	0.9973	0.644	0.9926	0.548	0.9963	0.693	0.9905	0.498	0.9978	0.572	0.9955
0.5	0.625	0.9961	0.768	0.9894	0.663	0.9946	0.821	0.9867	0.609	0.9967	0.690	0.9935
0.6	0.727	0.9947	0.869	0.9865	0.765	0.9929	0.922	0.9832	0.711	0.9956	0.792	0.9914
0.7	0.818	0.9933	0.946	0.9840	0.851	0.9912	0.992	0.9806	0.802	0.9944	0.876	0.9895
0.8	0.893	0.9920	0.993	0.9824	0.920	0.9897	1.029	0.9791	0.881	0.9932	0.939	0.9879
0.9	0.955	0.9908	1.011	0.9817	0.970	0.9885	1.031	0.9790	0.948	0.9921	0.981	0.9868
1.0	1.000	0.9899	1.000	0.9821	1.000	0.9878	1.000	0.9802	1.000	0.9912	1.000	0.9863

Calculated values of ρ and T_1

Figure 3 shows how they coinside with the experimental values listed above, and the agreement affirms us that our "vacuum solution" can afford fairly good approximation whenever the parameter η and γ lies under certain critical values. If we concern with the yarn tension, it will be reckoned from equation $T_1 = 1 - 2c^2\eta^2/a^2 \cdot \rho^2$.
Table 7 gives the calculated values of ρ and T_1, with the initial angle α_0 for the "non-licking case".

Next we must examine the "licking" cases. Of course the vacuum solution obviously can not give such "licking" figures, so we must turn back to the orignal equation stated in section 1, in which the air drag term is fully contained. Accoding to rough estimation of the air drag coefficient P, calculation should be repeated in varying the constant α and γ, till the good argreement between the theory and measurement is obtained.

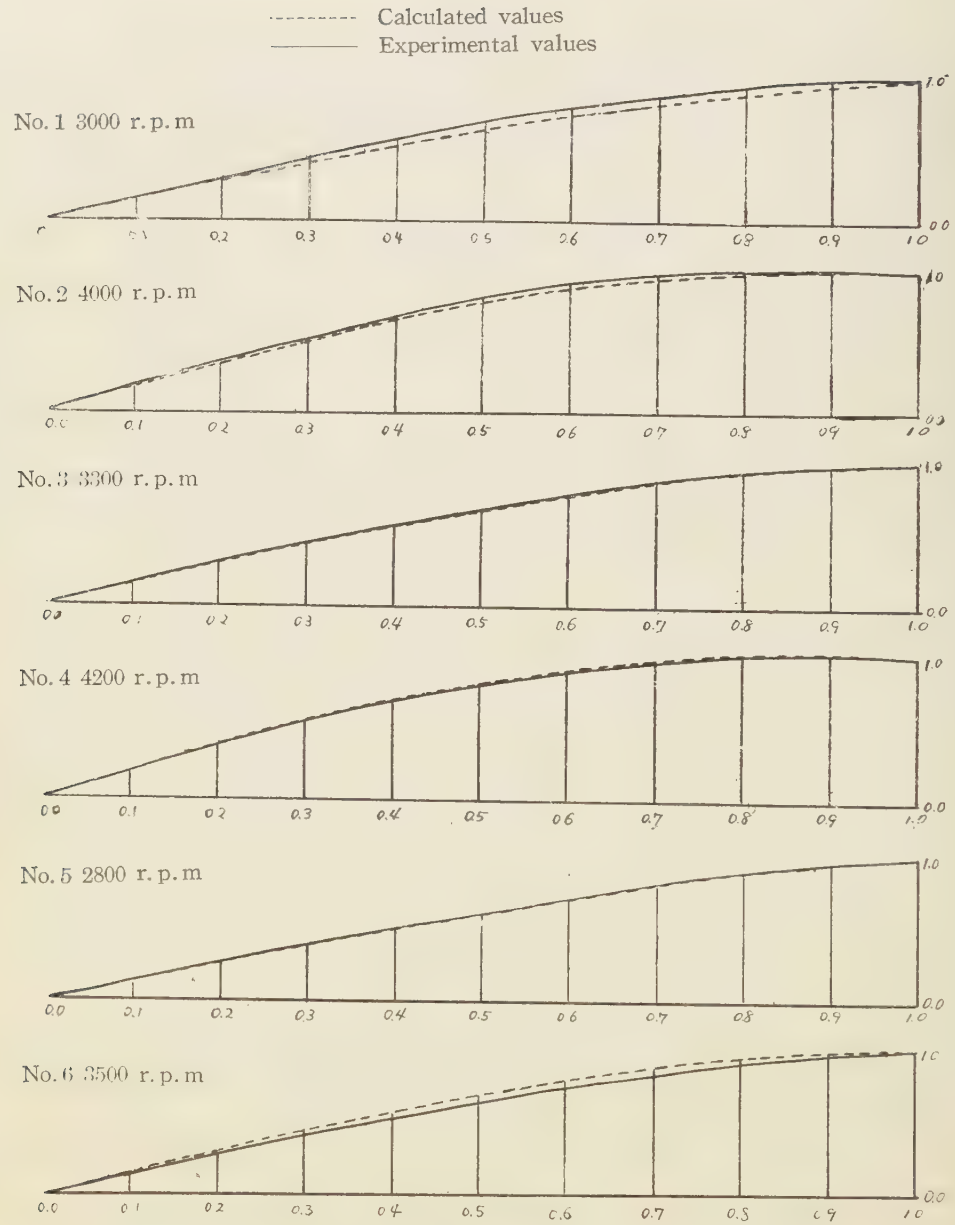


Fig. 3

Following Table 8 gives the results of calculation for the case No. 10 done in this way, and one sees its agreement is also fairly good. Main difficulty of the problem is how

to choose the suitable α_0 assuming the value of η. The value of γ calculated from the eq (1. 15) has the tendency to be slightly too small. For example of No. 10 we must take $\eta=2.4$ in place of 2. 01.

Theoretical increment of θ will be easily calculated independently from the equation (1. 13) using the calculated values of T_1, $\cos\tau$, and ρ. $\theta theor.$ thus obtained is 207° against the $\theta experim.=208°$. However, it is infered that the θ will be affected by the capedge friction rather deeply, therefore, we must be content with only to use this equation as a after proof of the solution.

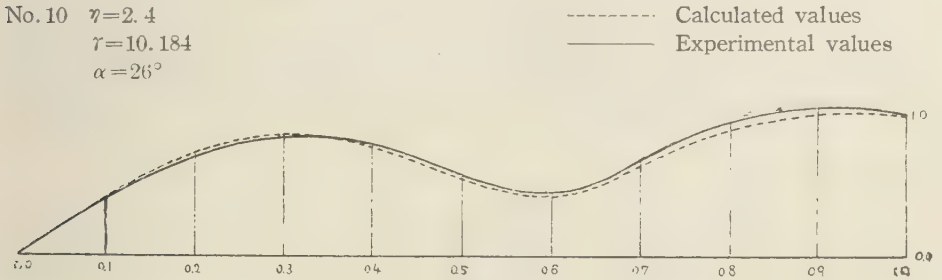


Fig. 4

6. Discussion and Conclusion

It is interesting to note that, accoding to Table 4, 5, 6, "Licking" does not arise if the value of η lies under about 1,00, and do arise if η lies above this critical value. Our presumption that our equations contain only the paremeter η and γ as the physical quantity and consequently the shape of ballooning shall be fully determined by these parameters, is quite in favour with the experiments. If the values of η satisfies $\eta \leqq 1,00$ and $\gamma \leqq 5.0$, no "licking" will arise and "vacuum solution" can be used for the explanation of shapes. In that case the physical quantity, such as the T_0, will be easily obtained by using the formulae stated in the section 2. There might be some capedge friction at the point $Z=1$, as J. Crank and D. D. Whitmore* pointed out, which should be disussed together with the motion of thread along itself. I shall discuss these points later, but it should be noted here that such additional friction has the tendency to increase

Table 8

No. 10 $\eta=2.4$
$\gamma=10.1849$
$\alpha_0=26°$

Z	ρ	T_1	θ
0.0	0.0000	1.0000	0°
0.1	0.4231	0.9751	0.°4
0.2	0.7314	0.9266	2.°1
0.3	0.8549	0.9051	7.°1
0.4	0.7595	0.9388	17.°7
0.5	0.5292	0.9933	42.°6
0.6	0.4201	1.0173	95.°1
0.7	0.6426	0.9964	150.°7
0.8	0.8869	0.9660	160.°8
0.9	1.0015	0.9712	194.°0
1.0	0.9965	1.0279	207°.5

Calculated of ρ. T_1. and θ

* Text. Res. J. (1954) Vol. 24 434

somewhat the values of η and γ. That is why we should have taken the value of η slightly larger than the theoretically predicted one.

Now in the example of No. 8 and 9 new phenomenon arise, which we will call provisionally as "flopping". That is, the ballooning thread changes its shape rapidly and periodically and does not stay in a stationary state. This comes from chiefly the irregularity of the rotational speed ω, and $\omega's$ change causes the variation of the $\eta-\alpha_0$ relation which must be satisfied in order to fulfil the boundary condition of the thread. At some critical value of ω, allowance range for ω is so narrow that a little fluctuation of ω ceases the motion to stay in the first "Eigen" state, just like the transition of electronic state from lower energy state to the higher energy state in the atomic physics.

At the present stage we cannot anticipate in which ω the flopping will arise, unless whole program of integration is complete. But I think it is very important to settle the matter decidedly for the practical purpose and our later efforts will be dedicated to this point.

Acknowlegement

For many different kinds of aid in the preparation of the experiments we are indebted to Mr. S. Irie & Mr. H. Kôzuma. Further we wish to express our thanks to Mr. Kamiya, who allowed us to use the materials, and gave helpful suggestions and advices.

SOME STUDIES ON VOLCANO ASO AND KUJIU (Part 10), GEOPHYSICAL STUDY OF ARIZONA METEORITE CRATER.

Munetosi NAMBA

(1954 April 1st.)

Abstract.

Analysis of the shape of Arizona Meterite Crater has brought some new facts to light. The incoming meterite rushing into the ground level in the direction of w-30°-N at an angle of incidence, 75°, and forming the crater of 580 meters in radius, explosed at the place of 300 meters deep; and the meteorite rock fragments were dispersed within a radius of 1010 meters.

1. Foreword.

Anylysing the shape of the depression caldera of Volcano Aso, the writer has been able to rationate the process of depression and understand distribution of central cones. [1]

Table 1. Radius vector for the Meteorite Crater.

No. of Radius vector	l_0 (E)	l_1	l_2	l_3 (N)	l_4	l_5
Observed Values	cm 1.90	1.82	1.90	1.80	1.80	2.12
in meters	m 570	546	570	540	540	636
No. of Radius vector	l_6 (W)	l_7	l_8	l_9 (S)	l_{10}	l_{11}
Observed Values	2.05	2.00	2.00	1.90	1.90	cm 1.95
in meters	615	600	600	570	570	m 585

Arizona Meteorite Crater exploded for the subterranean intrusion of the meteorite, and formed a great excavation. Then, analysis of the shape of Arizona Meteorite Crater will make the difference between Volcano Aso and Arizona Crater clear.

2. Result of Analysis.

As the writer did not happen to have on hand a detailed map of Arizona Meteorite Crater, he calculated referring to the map drawn to a scale 1/30,000 at page 240 of "Out of the Sky" by Dr. H.H. Nininger.

A reference point 0 was taken in the center of the crater. Radius vector from the

point 0 to the east rim crest was taken for l. Each of l_0, l_1,..., and l_{11} was calculated, measuring the radius vector, counter clock wise, from the point 0 to the rim crest every 30°. (Table 1)

Maximum distribution of the meteorite rock fragments ejected from the crater is shown on the lower limit of the said map; and the radius vector corresponding to it, $L_0, L_2, L_3, \ldots$, and L_{11} were calclated. (Table 2)

Table 2. Radius vector for the Lower Limit.

No. of Radius vector	L_0 (E)	L_1	L_2	L_3 (N)	L_4	L_5
Observed Values	cm 2.82	4.30	3.35	2.95	2.70	2.70
in meters	m 846	1290	1005	885	810	810
No. of Radius vector	L_6(W)	L_7	L_8	L_9(S)	L_{10}	L_{11}
Observed Values	2.70	3.10	3.60	3.40	3.60	cm 3.15
in Meters	810	930	1080	1020	1080	m 945

As a result of harmonic analysis, the following facts have come to light.

(1) Equation for the crater rim:-

$$l = 1.928^{cm} + 0.081\ cos\ (\theta - 201) + 0.067\ cos\ (2\theta - 354) + 0.084\ cos\ (4\theta - 259) + \cdots\cdots \pm 0.022\ (1/30.000)$$

or

$$l = 579^{m} + 24^{m}\ cos\ (\theta - 201) + 20\ cos\ (2\theta - 354) + 25\ cos\ (4\theta - 259) + \cdots\cdots \pm 6.6^{m}$$

(2) Epuation for th lower limits:-

$$L = 3.36_4^{cm} + 0.65_4\ cos\ (\theta - 330) + 0.44_0\ cos\ (2\theta - 134) + 0.21_5\ cos\ (3\theta - 112) + \cdots\cdots \pm 0.23_3\ (1/30.000)$$

or

$$L = 1009^{m} + 196^{m}\ cos\ (\theta - 330) + 132\ cos\ (2\theta - 134) + 65\ cos\ (3\theta - 112) + \cdots\cdots \pm 70^{m}$$

On depression crater, usually, the term suggesting a dimond type of a fissure is calcuted clearly, but, in this case, it is neglisible small. This may show the fact that the meteorite crater was formed by entirely external cause. Surmising from the above result of analysis the meteorite crater is an ellipse much the same as a circle with major axis

in the direction from W-30°-N to S-30°-E. (Fig. 1).

If so, it is sure that Arizona Meteorite Crater was formed by an incoming meteorite rushing into the ground from the direction of north-west along the major axis. And the center

Fig. 1. A. Sketch map of Meteorite Crater by Nininger

of the crater lies, along major axis, 0.08 (24 meters) north-west of the reference point 0.

The crater is much the same as a circle of which radius is 1.93 (530 meters). The lower limit of the ejected meteorite rock fragments has its center, along major axis, 0.654 (196 meters) south-east of the reference point 0, and it is much the same as a circle with a radius of 3.364 (1010 meters). An angle of incidence of the incoming meteorite, presuming from the ellipticity, is about 75° to the level.

Therefore, the gist of the matter is given as follows. Arizona meteorite rushing into the ground from the direction of W-30°-N at about 75°, and forming a crater of 580 meters in radius, an ellipse much the same as a circle, exploded under the ground; and the ejected meteorite rock fragments were dispersed within a radius of 1010 meters.

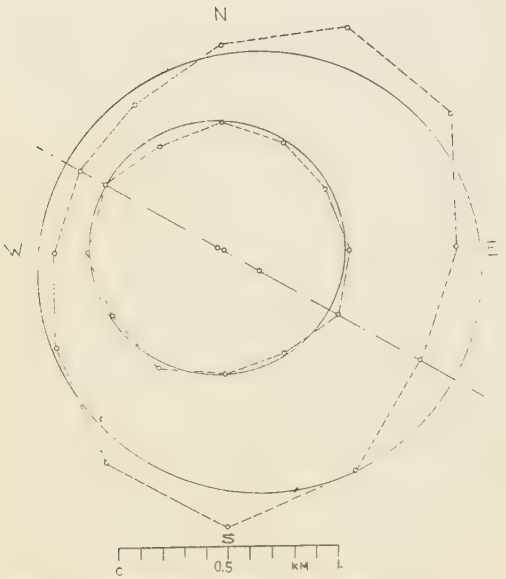


Fig. 1. B Calculated crater and Lower Limit. dottedline: -calculated values

But a distribution circle of the meteorite rock fragments was deviated 0.08-0.645 (220 meters) off along the major axis for inrush due to the evacuated trail. Now the figure of a vertical section of Arizona Meteorite Crater, based on the data by Dr. H. H. Nininger, is as follows. (Fig. 2) Mean height from the ground level to the rim crest is 186 ft. (60 meters) and the ejected fragments amount to

$$\pi/3\times 1.0\times 10^8.\ m^3.$$

Supposing these fragments to be ejected with explosion from the pit on the top of the cone with the base of 580 meters in radius, the pit is 300 meters deep under the ground leyel.

This, on the whole, concides with result of drilling by the Survey of International Geophysics in 1930, that showed the subterranean meteorite lies about 990 ft. (330 meters) in depth under the ground,

Maximum horizontal velocity must be about 84 m/sec. (neglecting the atmospheric resistance) to project the meteorite fragments on the ground level within a circle of 1010

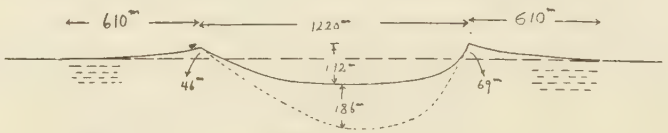


Fig. 2. Section of Meteorite crater. Looking eastward.

meters in radius perforating through the crater 300 meters deep under the ground. Effect of inrush due to the evacuated trail of incoming meteorite deviates the center of the projection circle from that of the crater by 220 meters, and, therefore, the effect of inrush must be 220×84/1010 m/sec, i.e., 19m/sec..

3. Conclusion.

1). It stands to reason that Arizona Meteorite Crater formed by external incoming intrusion, contrary to depression caldera, has no fissure caused by depression of the ground.

2). Configulating of Arizona Meteorite Crater at pressent was formed by the explosin of

the meteorite due to the subterranean intrusion.

3). Dr. H. H. Nininger, on his result of observation, has held that many meteorite fragments was dispersed on the lines of major and minor axis of the crater (p. 240 Fig. 21). The writer can understand the distribution of many meteorite fragments on the lines of major axis, but the fine fragments on the line of minor axis are above his comprehension; this demands the further study in future.

4). Survey of Arizona Meteorite Crater threw light upon the fact that the subterranean meteorite consists of two concentrations of fragments (Out of the Sky). And, ever since, the two concentrations of fragments have been believed to rush into the ground almost the same at the very same place. The writer has no materials to criticixe the problem. But the writer has observed the formation of volcanic bombs for years; and, by the long course of experience, he wishes to make a suggestion.

The writer has often observed the upper (half) of a plastic volcanic bomb falling down on the ground slide forward for its high speed. And, under such cifcumstances, they seem as if they were two bombs falling down almost the same time at the very same place. It is a rare occurrence that two meteorites rush into the ground almost the same time at the very same place. The writer can form no idea of the scene, unless he sees it for himself. There are very few, if any, such cases.

In the writer's opinion, the upper halh of an incoming meteorite slided foreward for its high speed, and the subterranean meteorite was devided into two concentrations of fragments. This is the reason why they seem as if they were two meteorites rushing into the ground almost the same line at the very same place. The writer feels this must be more probable case.

Litterature.

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On the shape of old Aso crater and the distribution of central cones and hot springs: - Kumamoto Journal of Science, Vol·1, 1952, pp. 66-73.
(2) H. H. Nininger: - Out of the SKy, p. 213.
(3) H. H. Nininger: - Ditto (2), p. 213-
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ON THE SURFACE FIGURES OF THE EARTH, THE MOON AND MARS. (REPORT III)

Saemon Taro NAKAMURA

Received Sept. 30 1954

Chapter IV. Harmonic series representing the figure of the whole earth.

As it is not sufficient to represent the surface figure of the whole earth, if harmonic terms up to the 3rd. order only are taken, those up to the 7th. order are calculated using all date over the whole earth. The expression is

$$h = \sum_{m=0}^{7} A_{0m} P_m(\mu) + \sum_{m=1}^{7} \sum_{n=1}^{m} \{A_{nm} \cos n\varphi \, P_m^n(\mu) + B_{nm} \sin n\varphi \, P_m^n(\mu)\} \quad \cdots (4)$$

The coefficients computed are as shown in the Table 4.

Table 4

n \ m	0	1	2	3	4	5	6	7
0	A	−0.4214	0.9322	0.6568	−0.3854	0.5737	−1.2965	0.4485	−0.7531
1	A		0.3791	0.1312	−0.1250	−5.112 ×10⁻²	−1.918 ×10⁻²	0.055 ×10⁻²	−2.468 ×10⁻²
	B		0.2783	0.1534	0.0150	−3.612 ×10⁻²	−2.409 ×10⁻²	−2.653 ×10⁻²	2.638 ×10⁻²
	$\sqrt{A^2+B^2}$		0.4703	0.2019	0.1259	6.257 ×10⁻²	3.079 ×10⁻²	2.654 ×10⁻²	3.612 ×10⁻²
2	A			−7.077 ×10⁻²	−5.218 ×10⁻²	−2.889 ×10⁻²	2.913 ×10⁻³	−2.818 ×10⁻³	4.695 ×10⁻³
	B			−2.342 ×10⁻²	6.933 ×10⁻²	0.819 ×10⁻²	1.174 ×10⁻³	2.129 ×10⁻³	3.825 ×10⁻³
	$\sqrt{A^2+B^2}$			7.454 ×10⁻²	8.677 ×10⁻²	3.003 ×10⁻²	3.141 ×10⁻³	3.532 ×10⁻³	6.056 ×10⁻³
3	A				0.656 ×10⁻²	7.692 ×10⁻³	1.063 ×10⁻³	0.515 ×10⁻³	−1.894 ×10⁻⁴
	B				2.882 ×10⁻²	−2.059 ×10⁻³	1.850 ×10⁻³	1.584 ×10⁻³	1.148 ×10⁻⁴
	$\sqrt{A^2+B^2}$				2.956× 10⁻²	7.963 ×10⁻³	2.134 ×10⁻³	1.666 ×10⁻³	2.215 ×10⁻⁴
4	A					−0.381 ×10⁻³	1.594 ×10⁻³	1.522 ×10⁻⁴	−1.348 ×10⁻⁴
	B					5.012 ×10⁻³	−0.570 ×10⁻³	1.600 ×10⁻⁴	0.503 ×10⁻⁴
	$\sqrt{A^2+B^2}$					5.026 ×10⁻³	1.693 ×10⁻³	2.208 ×10⁻⁴	1.439 ×10⁻⁴
5	A						0.582 ×10⁻⁴	−1.563 ×10⁻⁵	7.271 ×10⁻⁶
	B						2.609 ×10⁻⁴	−7.443 ×10⁻⁵	−4.832 ×10⁻⁶
	$\sqrt{A^2+B^2}$						2.673 ×10⁻⁴	7.597 ×10⁻⁵	8.730 ×10⁻⁶
6	A							6.465 ×10⁻⁶	−0.899 ×10⁻⁶
	B							1.141 ×10⁻⁶	−3.036 ×10⁻⁶
	$\sqrt{A^2+B^2}$							6.565 ×10⁻⁶	3.166 ×10⁻⁶
7	A								−0.270 ×10⁻⁷
	B								−4.452 ×10⁻⁷
	$\sqrt{A^2+B^2}$								4.460 ×10⁻⁷

It is remarkable that A_{05} is very large. It is largest of all coefficients. It makes us to think of some agency with a frequency corresponding to the mode given by this term

$P_{05}(\mu)$. Lord Kelvin[1] gave the frequency of the free oscillation of liquid globe. According to him, in the case of uniform sphere, the frequency σ_n of the oscillation represented by zonal harmonics of the n th. order is given by

$$\sigma_n^2 = \frac{2n\ (n-1)}{2n+1}\ \frac{4}{3}\ \pi\gamma\rho, \qquad \cdots\cdots (5)$$

where γ is the gravitational constant and ρ the density of the globe. In the case of the earth ρ is 5.52. σ_n computed by (5) and corresponding amplitudes in (4), are given in Table 5.

Table 5.

n	1	2	3	4	5	6	7
$\sigma_n \times 10^4 (\mathrm{sec}^{-1})$	0.0	11.1	16.2	20.3	23.7	26.7	29.4
A_{0n}	0.9322	0.6568	− 0.3854	0.5737	− 1.2965	0.4485	− 0.7531

The coefficients for the moon, about 60 % of which is known, will give some information of the physical meaning of the relation between the period and amplitude. It may be also interesting to know of the coefficients for mars, which is the unique one of all planets whose surface figure is known.

1) W. Thomson: Dynamic problem regarding elastic spheroidal shells and spheroids of incompressible liquid. Phil. Trans. 1863. also see Love; Hydrodynamics 8th. edd. p. 450.

SOME STUDIES ON VOLCANO ASO AND KUJIU (PART 11)
THE FLOOD OF THE RIVER SHIRAKAWA ON JUNE 26-27, 1953

(THE 2nd REPORT)

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Department of Physics, Faculty of Science, Kumamoto University

(Received September 30, 1954)

Abstract

(1) The writer estimated the quantity of water which overflowed in Kumamoto City in the case of the flood of the River Shirakawa which took place on June 26-27, 1953, (2) he also observed the waterlevel and the current velocity of the Shirakawa at Ozeki-bridge (the east end of Kumamoto City) and other points along the river in the rainy season, namely June and July 1954. Making use of the results (1) and (2), he studied the relation of the water-level to the current velocity and the rate of flow at Ozeki-bridge.

1. The Quantity of Flow in the Case of the Flood in 1953

As mentioned in the previous paper [1], in the case of the flood in 1953 muddy water overflowed in Kumamoto City, and after the flood had been over, there deposited a large quantity of mud. Our field study and calculation found out the following values for Kumamoto City, (1) the flooded area about 35 km^2, (2) the area covered with mud about 12 km^2, (3) the apparent volume of the deposited mud about $2.5 \times 10^6 m^3$, and (4) the mean height of the deposited mud about 20cm.

Owing to our researches, the rate of mud to the muddy water is about 5%, and the porocity of the deposited mud is about 55% just after the flood. Judged from the state of the flood in Kumamoto City, it may be probably assumed that about 80% of the mud which had been contained in the muddy water deposited in the city. Then we get the total quantity of water which overflowed in Kumamoto City, as follows

$$(2.5 \times 10^6 \times 0.45) \div (0.05 \times 0.8) \fallingdotseq 28.1 \times 10^6 \ m^3 \quad \cdots\cdots (1)$$

In the case of the flood, the River Shirakawa broadly over-flowed near Ozeki-bridge which was washed away by the flood about $17^h\ 40^m$ on 26th, and the river flooded Kumamoto City. Therefore an investigation about the state of flow at Ozeki-bridge will tells us the flood discharge which flowed through Kumamoto City. The secton of the River Shirakawa at Ozeki-bridge is shown in Fig. 4.

In usual case, if the water-level at Ozeki-bridge (the reading of staff-gauge) overs about 4.5m, the River Shirakawa will begin to overflow in Kumamoto City. But, in the

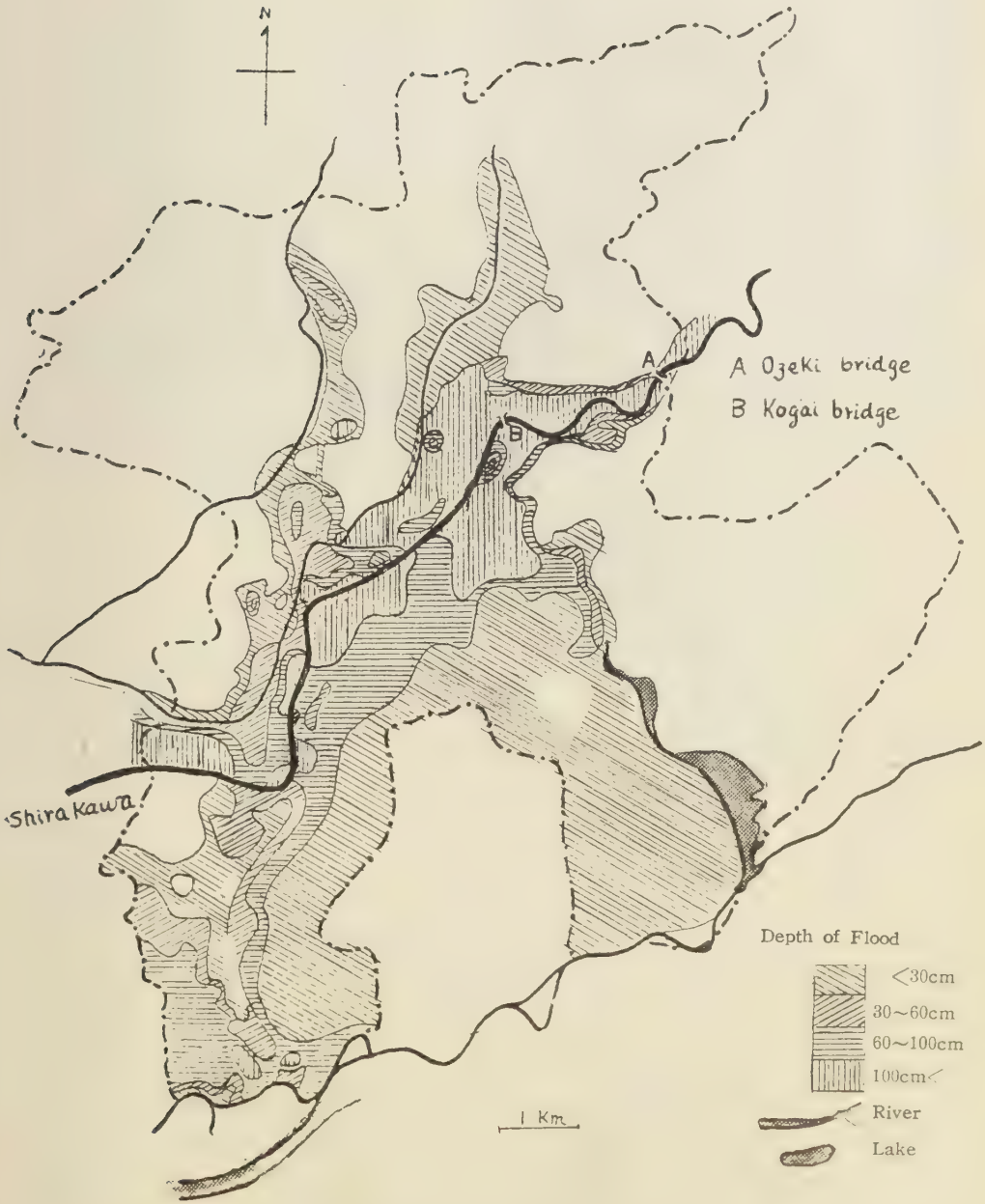

Fig. 1. Flooded Area in Kumamoto City

Fig. 2. Area covered with Mud in Kumamoto City

case of the flood on June 26-27, 1953, immense quantity of drift wood was caught by Kogai-bridge (about 2.8 km down the river from Ozeki-bridge), and the river channel became narrow here, consenquenty the water-level remarkably rose up at the upper part of this bridge. [2] And at the peak of the flood Ichiya-tomo (the right bank about 100 m up the river from Kogai-bridge) collapsed, and after a short while the left bank of the river was broken at Kogai-bridge. Judged from the facts above descrived, we may consider that in this case total quantity of water which flowed through the part of water course above the level 4 m at Oze-

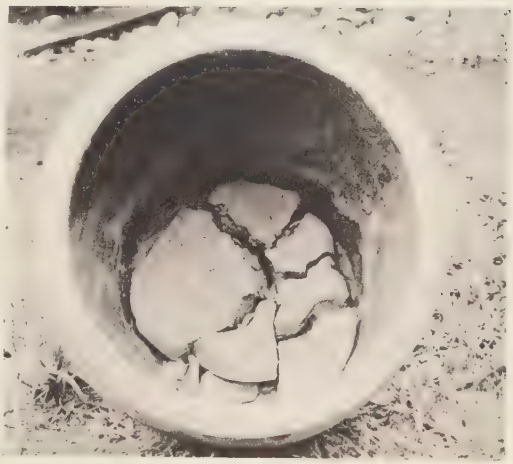

Fig. 3. Mud deposited in a Jar (Photographed about a Year after the Flood)

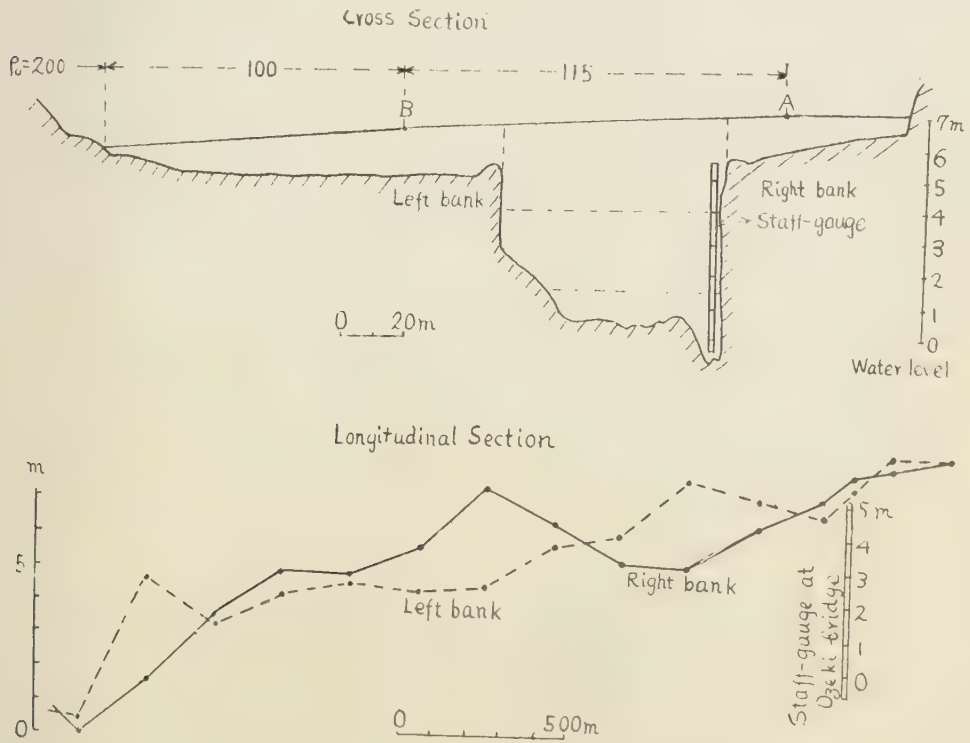

Fig. 4. Section of the River Shirakawa at Ozeki-bridge

ki-bridge overflowed in Kumamoto City. Floodmarks show that the sectional area of water course above the level 4 m was about 400 m² at the peak of the flood, and water-level was higher than 4 m for 10 hours. Now, equating the total quantity of water which flowed above the level 4 m for 10 hours to the quantity of water previously deduced from the deposited mnd, we get rouguly

$$(400 \times 60^2 \times 10V) \div 2 = 28.1 \times 10^6$$

where V is 10 hours average of the mean current velocity of flow which flowed above the level 4 m at Ozeki-bridge,

then $V = 3.9$ m/sec ..... (2)

Making use of the height of flood mark at A on the right bank and at B on the left bank near Ozeki-bridge (C.F. Fig. 4), by the formula which gives approximate value of water-level difference

$$y = \frac{v^2}{g} log\left(1 + \frac{x}{\rho_0}\right)$$

we get $$0.85 = \frac{2.3}{9.8} v^2 \left\{log_{10}\left(1 + \frac{215}{200}\right) - log_{10}\left(1 + \frac{100}{200}\right)\right\}$$

then $v = 5.07$ *m/sec.* ..... (3)

This is an approximate value of the mean current velocity at Ozeki-bridge at the peak of the flood.

Next, using the value of slope 0.0032 whih was gotten from the flood marks, and dividing the water course into two parts, namely river channel and water channel over both banks, we calculated the mean current velocity in two parts at Ozeki-bridge by Kutter-Chézy formula and got the following values as shown in Table 1.

Table 1

Part	Hydraulic rad. (m)	Roughness coeff.	Mean vel. (m/sec)	Sectional area (m²)
River channel	5.7	0.030	5.9	430
Part over both banks	1.05	0.035	1.7	200

From these values, we get approximately

the mean current velocity at the peak $= (5.9 \times 430 + 1.7 \times 200) \div 630 = 4.6$ m/sec ..... (4)

This value should be somewhat small, because we calculated as if the boundary of the two parts was fixed. Taking the fact above described and the state of the flood into consideration, from (3) and (4) we adopt 5.0 m/sec for the mean current velocity at the peak of the flood in 1953. Then the maximum dischrge becomes

$5.0 \times 630 = 3150$ m³/sec ..... (5)

2. Freshet in the Rainy Season, 1954

We observed the water-level, the current vlocity and the slope at Ozeki-bridge, in the rainy season, 1954, namely June and July. The results are shown in Fig. 5, and some values are tabulated in Table 2. The water-level is the reading of the staff-gauge at Ozeki-bridge and it varied in range from 1.5m to 3.9m during our observation. Hence the values in range from 3.9m to 5.0m were calculated, taking the state of the river channel into consideration.

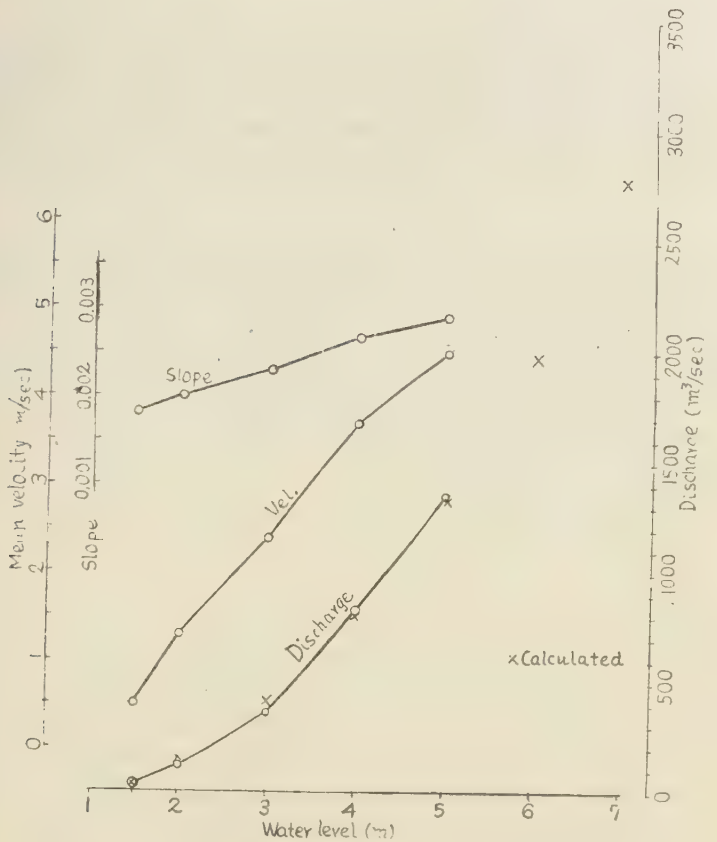

Fig. 5.

The emperical formula for discharge-stage curve are gotten as follows

$$Q = -62 - 29H + 62H^2 \quad \cdots\cdots (6)$$

where Q = quantity of flow (m^3/sec) at Ozeki-bridge

H = water-level at Ozeki-bridge

$1.5\ m \leqslant H \leqslant 5.0\ m$

Table 2

Water level (m)	Slope	Mean velocity (m/sec)	Discharge (m^3/sec)
1.5	0.0018	0.5	34
2.0	0.0020	1.3	123
3.0	0.0023	2.4	384
4.0	0.0027	3.7	840
5.0	0.0029	4.5	1350

On probation we calculated Q using (6) to the level 7.0 m and the results are shown in Table 3. The difference between the calulated value Q=2770m^3/sec for H=7.0 m and the maximum discharge 3150 m^3/sec in 1953 should be mostly attributed to the fact that in real case when the water-level goes up higher than 4.5~5.0 m the river begins to overflow.

Table 3

Water-level (m)		1.5	2.0	3.0	4.0	5.0	6.0	7.0
Discharge (m^3/sec)	Value in Table 2	34	123	384	840	1350		
	Calculated	34	I28	409	814	1343	1996	2773

If we assume that the value 3150 m^3/sec is probable, we have the modified equation for Q as follows

$$Q = (-62 - 29H + 62H^2)\{1 + 0.07\ (H - 5)\}\ m^3/sec \quad \cdots\cdots (7)$$

where 5.0 m<H.

4. Conclusions

The main points of the present studies are summarised as follows.

(1) In the case of the heavy flood which was experienced in Kumamoto City on June 26-27, 1953, flooded area was about 35 km^2, the quantity of water which overflooded the city was about $28\times10^6 m^3$ and about $2.5\times10^6 m^3$ of mud deposited there over the area about 12 km^2. The water-level at Ozeki-bridge was higher than 4.0 m for 10 hours, and total quantity of water which flowed above the level 4.0 m motsly overflooded Kumamoto City and there deposited the quaatity of mud above mentioned.

(2) The emperical formula for discharge-stage curve at Ozeki-bridge is as follows

$$Q = -62 - 29H + 62H^2 \qquad 1.5\ m \leqslant H \leqslant 5.0\ m$$

$$Q = (-62 - 29H + 62H^2)\{1 + 0.07\ (H - 5)\} \qquad 5.0\ m < H$$

where Q in m^3/sec, H in m.

In regards to the relation of the rain-fall in catchment area to the water-level and the discharge, the writer will report in the following paper touching the problem of flood pre-

diction. This work was carried out under the support of the Scientific Research Expenditure

In concluding this paper the writer wishes to express his hearty thanks to the following persons; to Prof. S. T. Nakamura and Prof. M. Namba for their kind guidance and encouragement, and to Mr. K. Doke and Mr. Y. Sugimoto for their help in observation and calculation.

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PAPER CHROMATOGRAPHY OF INORGANIC IONS BY USING ORGANIC ANALYTICAL REAGENTS. I.

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(With 3 Tables)

Received September 29

Introduction

Most of the work on paper chromatography done to date has been of a partition chromatography between the adsorbed water on the filter paper (fixed phase) and the wash liquid (mobile phase). Subsequently, the water-insoluble samples were brought almost to the front of development and their separation was almost impossible, even if the wash liquid was selected to solve the samples. The author wished to construct the chromatographic systems, which had the freedom to choose substances of fixed phase in some extent. These systems may bring some hope to the separation of water-insoluble samples. The experiments were undertaken to combine the alumina impregnated filted paper which adsorbed the organic solvents and some proper wash liquid, so that some partitions might occur between the adsorbed phase (fixed phase) and wash liquid phase (mobile phase). The samples were of the chelate dithizone compounds, which were formed by the reaction under the same condition between various kinds of inorganic ions and dithizone, and the compounds were almost insoluble in water (1).

Experimental

A sheet of filter paper (Toyo filter paper No. 5B) was allowed to stand completely immersed in the saturated solution of alum ammoniacal for one hour at room temperature. then it was taken out and most of the superfluous solution was drained off. For the following one hour, the moist filter paper was hanged in the ammonia vapour. It was then washed in the stream of water and dried up at 110°C. in the electric drier. Successively, the dried impregnated paper was placed in the atmosphere of a closed vessel saturated with the vapour of the organic solvent, which was desired to be adsorbed on the filter paper. After leaving 24 hours in the closed vessel, the filter paper was allowed to use in the following experiments.

The treeted filter paper was cut into 1×20 cm. strips. The samples were prepared by adding an excess of carbon tetrachloride saturated by dithizone to the aqueous solution of the ions to be investigated. The metal chelates formed were contained in the coloured lower layer (carbon tetrachloride layer), and the solution itself was used as spotting sample.

Strips of treated filter paper were spotted with the samples, then hanged in a glass

chamber containing the desired solvent and developed by the same procedure as the usual paper chromatography.

Table I. R_f Values of Dithizonates in the System of Carbon Tetrachloride as both Fixed Phase and Mobile Phase.

Metal	Complex E : Enol form K : Keto form	Colour	R_f	Remarks
Bismuth	E	Orange	0.19	
Cadmium	K	Orange-red	0.00	Streaking from the starting point
Cobalt	K	Violet	0.30	
Copper (II)	E	Yellow-brown	0.94	
Mercury (II)	K	Orange-yellow	0.75	
Nickel	....	Violet	0.40	
Silver	E	Violet	0.00	
Zinc	K	Purplish red	0.00	Considerably streaking from the starting point

Table II. Dependence of R_f on Alcohol Chain Length.

Metal	R_f Values		
	Methnaol	Ethanol	N-Butanol
Bismuth	0.69	0.91	0.92
Cobalt	0.82	0.91	0.92
Copper (II)	0.50	0.52	0.82
Mercury (II)	0.61	0.87	0.90
Nickel	0.65	0.79	0.92
Zinc	0.00	0.88	0.80

Table III. R_f Values of Dithizonates in the System, Chloroform as Fixed Phase and Carbon Tetrachloride as Mobile Phase.

Metal	Complex E : Enol form K : Keto form	Colour	R_f	Remarks
Bismuth	E	Orange	0.60	
Cadmium	K	Orange-red	0.00	Streaking from the starting point
Cobalt	K	Violet	0.61	
Copper (II)	E	Yellow-brown	0.87	
Mercury (II)	K	Orange-yellow	0.82	
Nickel	....	Violet	0.74	
Silver	E	Violet	0.00	
Silver*	K	Brown	0.00	Streaking from the starting point
Zinc	K	Purplish red	0.72	

*The sample was formed in dilute mineral acid solution.

Table I shows the results of developed chromatograms, both the fixed and the mobile phases were carbon tetrachloride. In these cases, there were no partition differences between the two phases (fixed and mobile phases), so the coloured zones of dithizonates were fairly diffused. Partition difference between the two phases may be effected to improve the separation of these dithizonates. Subsequently, the following studies were carried out.

Table II shows the R_f values of chromatogram, when the fixed phases were carbon tetrachloride and the mobile phases were alcohols. In these cases, there was a general tendency, except the zinc dithizonate, to increase the R_f values as the alcohol chain length increased. As the alcohol chain was decreased, the greater differences among the zones of dithizonates appeared, but on the other hand, zones were more or less spreaded. Anyway, it was inconvenient to use alcohols as mobile phase to separate these dithizonates.

Table III shows the results of the developments, when the fixed phases were chloroform and the mobile phases were carbon tetrachloride. These cases presented the best results in the trials I had ever made.

Conclusion

The partition chromatographic system in which both the fixed and the mobile phases were formed by organic solvents was more convenient to separate insoluble substances than the usual paper partition chromatography in which the adsorbed water in the filter paper was solely used as the fixed phase. To apply this chromatographic system to the field of analytical chemistry, I studied, as one of its examples, the chromatographic separation of dithizonates: the separation of dithizonates has usually been performed with a careful adjustment of pH values and a judicious use of masking agents.

Among the results of these trials, the most hopeful one was obtained by the system of chloroform (fixed phase) ——carbon tetrachloride (mobile phase), but the separtion of dithizonates were not completely established in this system, so the further studies are going on.

Literature

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INORGANIC ANALYSIS BY ELECTRO PAPER CHROMATOGRAPHY. I.

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(With 8 Figures)

(Received September 29)

Introduction

By the use of the filter paper for the supporter of the electrolytic solutions, various kinds of electromigrations have been reported to date. Especially in the field of the isolation of protains which was almost impossible by partition chromatography, many excellent studies have been done. In inorganic analysis this method has recently been introduced and applied principally on the analysis of the inorganic cations (1–7). In the analysis of the inorganic anions, there are few works. This paper attempts to elucidate on the electromigration of chlorine, bromine and iodine ions on the paper. And on the basis of the data obtained, the author found a method of effective separation of these three anions.

Experimental

Figure 1 shows the apparatus which was used for this experiment.

For supporting electrolyte, aqueous lactic acid was used. Lactic acid solutions are recommended as suitable conductors of the electrical current, by H. H. Strain *et al.* (6). Figure 2 shows the effect upon the migration of chlorine ion, bromine ion and iodine ion in lactic acid of different molal concentrations.

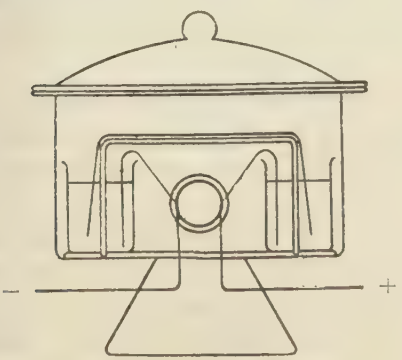

Figure 1. Apparatus for Development of Chromatogram.

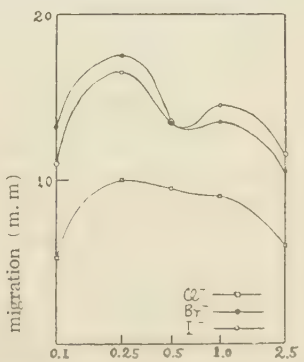

Figure 2. Migration of Chlorine, Bromine and Iodine ions during 10 Minutes in Lactic Acid of Various Concentration.

Toyo filter paper No.5B was used. A dry sheet of filter paper (4×28 cm.) was moistened with the lactic acid and after a few minutes the excess electrolytic solvent was removed with blotting paper. The moisted paper was placed horizontally on the glass frame which was coated with vinyl-acetal resin. Both ends of the paper was immersed in the electrolytic cells. In the cells, the lactic acid solutions were poured about 2cm. in depth, and 25c.c. in volume. The platinum wire electrodes were placed in the cells. One hour was permitted for equilibration of flow of the electrolyte through paper, then the solution of each ion to be investigated was added from sharp-pointed micropipet to the center of the paper sheet and 300 volt was applied to the electrodes from an electronic rectifier (Figure 3). The samples used contained potassium chloride, potassium bromide,

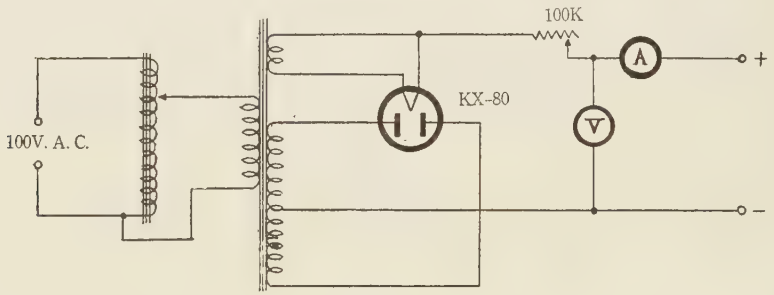

Figure 3. Wiring Diagram of Rectifier.

and potassium iodide each 10mg. in 1c.c. aqueous solution and on the paper added about 0.005 c.c. from the pipet.

After 10 minutes electrolysis, ions on moist paper were located by spraying aqueous silver nitrate solntion and sensitized under ultra-violet light.

Figure 2 shows that in the region of concentration of lactic acid less than 0.5 M., the order of migration velocity of these three halogen ions coincides with their mobility series, but in the region of concentration of more than 0.5 M., the migration velocity order of bromine ion and iodine ion is inversed.

At the concentration of 0.25 M., all the ions show their maximum migration, and at 1.0 M., there appeares a small peak in migration velocity. To elucidate the relationship between migration of ions and electrolyzing time, at these concentrations of electrolyte, the following experiments were done.

Figure 4 which is the case of maximum migration (concentration of lactic acid is 0.25 M.), shows that during 5 to 20 minutes the relation between these two factors (migration distance—time of electrolysis) are linear. It is plausible to asume that the ions, originally added at the center of the moist sheet of paper as aqueous solution, migrated from the aqueous region to that of lactic acid, that is, the velocity of ions increased gradually and after about 5 minute electrolysis, these ions were all in the region of lactic acid—in the concentration of maximum velocity (0.25 M.). The increase of velocity, later

than 20 minutes, would be due to the temperture rise through electrolysis.

Figure 5 shows the migration in 0.5 M. lactic acid. In this case the migration velocity of bromine ion and iodine ion were equal during the time of 10 to 20 minutes electrolysis.

Figure 6 shows that in 1M. lactic acid, iodine ion migrates faster than bromine ion from the begining to the end.

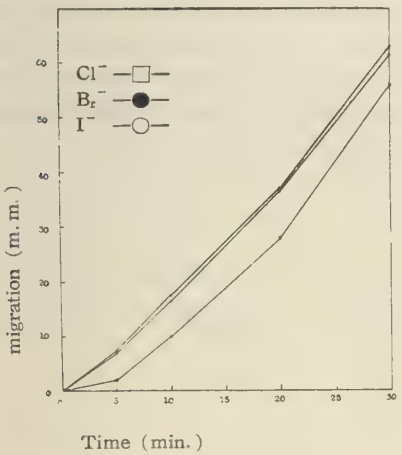

Figure 4. Migration of Chlorine, Bromine and Iodine Ions in 0.25 M. Lactic Acid with Time.

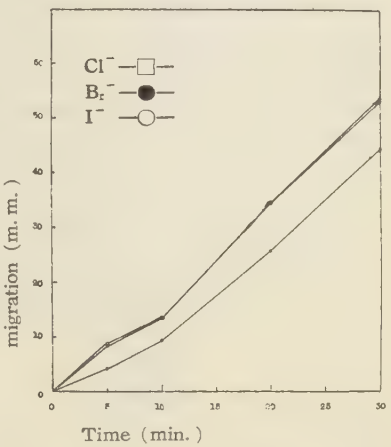

Figure 5. Migration of Chlorine, Bromine and Iodine Ions in 0.5 M. Lactic Acid with Time.

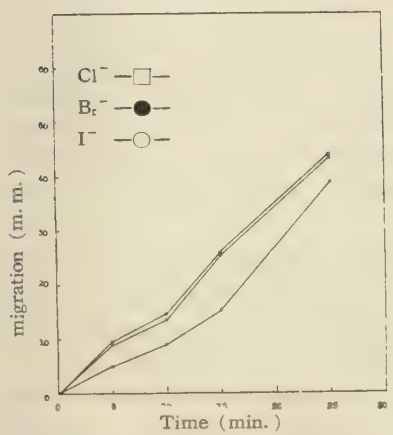

Figure 6. Migration of Chlorine, Bromine and Iodine Ions in 1.0 M. Lacttic Acid with Time.

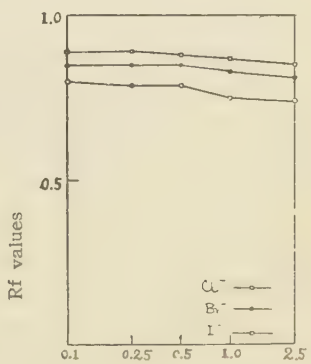

Figure 7. Influence of Concentration of Latic Acid upon R_f values of Chlorine Bromine and Iodine ions.

From the facts showed in figure 3 to 5, the migration time is preferable to decide

in 10 to 20 minutes. Moreover, the elongated electrolysis caused the diffuse spots of ions and the separation of ions was not improved at all. This is the principal reason that I prefered 10 miniutes as the migration time in figure 2.

The affinity between paper and these three ions was estimated by usual descending paper chromatogrphic procedure. Figure 7 shows the results of these experments: the influence of concentration of lactic acid upon R_f values of these ions. Comparing figure 2 with figure 7, the order of migration velocity of these ions are completely inversed in the region of concentration of more than 0.5 M. of lactic acid. It can be expected that when these two methods——chromatographs with and without electrolysis——are combined so as to make their effects upon ionic migration to be additive, the separations of these three ions must be more effective than when one of these is used alone.

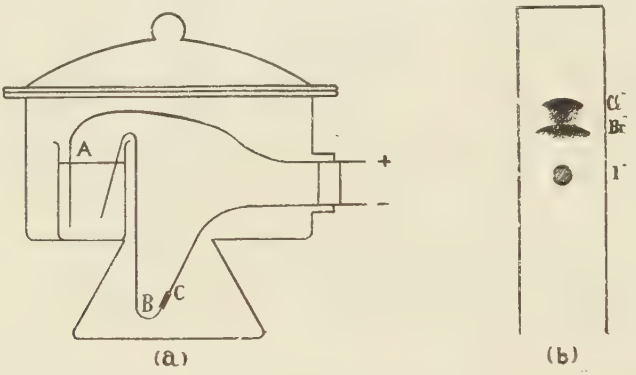

Figure 8.

Figure 8 (a) shows the apparatus of this experiment. (A) was the wash liquid cell and the platinum wire electrode was placed in it as the anode. One end of the dray filter paper (4×28 cm.) was immersed in the wash liquid in the cell. The other end of the paper was pressed firmly to the cathode (platinium wire) inside a bended platinium plate, The procedure was as follows.

At first, solutions of ions to be investigated, added on the paper in spot about 5cm. from the upper end of the paper, developed by the same procedure as the usual descending paper chromatography, and when the front of wash liquid reached the lowest portion of the paper (B), a small piece of filter paper fully contained with the wash liquid was attached to the portion of between (B) and (C). So that the liquid front and cathode plate were connected by the wash liquid. Then, between the anode and cathode, electrolytic current was able to be conducted. Figure 8(b) shows the result of above procedure: wash liquid was 1M. lactic acid, potential applied was 300 volts and the electrolytic time was 5 minutes.

Chlorine ion and bromine ion were separated but there was no branc space between

them. At the front of these ions, however, they were defined so clearly that it was easy to identify them. This well defined front of ions was one of the major points in electrochromatography. Between iodine ion and bromine ion there was sufficient space to separate them.

Conclusion

As to the three halogen ions, treated in this paper, the order of the velocity on the moist filter paper was inversed when electrolysis was employed or not. This fact means that the chromatographic behavior on the paper and the electromigrational behavior of these ions are compensative. So the elongated electrolysis caused the diffuse spots and no effective separtion was produced. Then, the separation of these ions will be fairly difficult by the usual electrochronatographic procedure—whether one dimensional or two dimensional are employed. So the usual paper chromatographic procedure was applied at first, and then the short time electrolysis was used so as to improve their separation and to give well defined fronts of ionic zones.

In concluding this paper the author wishes to express his hearty thanks to Prof. E. Ohara and Prof. K. Ochiai for their kind guidance and encouragement.

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DISTRIBUTION OF RADON IN CENTRAL KYUSHU (PART II)

Koichi KAKU

(Received September 30, 1954)

In continuation of the former report, I have discussed herein the spouting mechanism of hot springs as inferred from the radon content of hot springs, temperature of the springs, etc. Also seeking for the current velocity of hot spring water from the half life period of radon, I got approximately 15m/day for Hinagu Hot Springs.

Section 6. Radon Distribution at Hinagu Hot Springs.

1. *General Facts and Results of Measurements.*

As shown in Fig. 6-1 Hinagu Hot Springs is located about 46 Km. from Kumamoto City and may be classified as common salt spring. About 30 spouts are active at present and the water is pumped up in most places. Some of the self-spouting hot springs stop spouting when pumps are being used extensively in the mornings and evenings in other

Fig. 6-1 Location of Hot Springs (Hinagu and Hitoyoshi)

places. Therefore the number of constantly self spouting hot springs is very small. The depth of the hot springs are between 30m-90m, with the largest number included in about the 50m group. The temperature of spouting water lies between 41 C-48 C, and the most of them lies

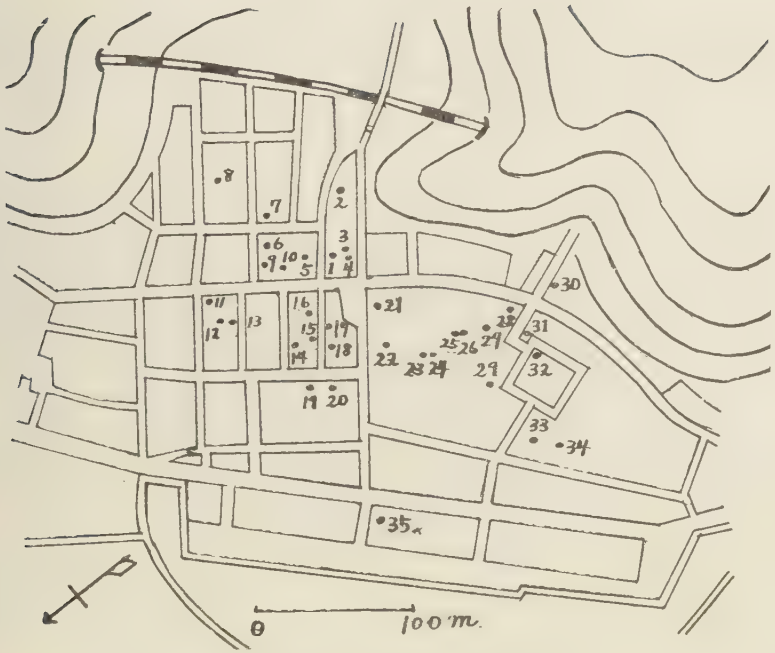

Fig. 6-2 Sites of Hot Springs.

at about 45°C. The distribution of hot springs is as shown in Fig. 6-2. Most of the hot springs are located at a distance of 100 meters or more from the beach. A large portion of this area is comprised of land reclaimed in Meiji-era, so since then not more than 50 years are passed. The method of measurement is as described in the previous report. Measurements were made on the spot during the period of October 15 to October 30, 1953. Results of measurements are shown in Table 6-1.

Table 6-1 Radon Content of Hinagu Hot Springs

No.	Springs	Ta	Ts	pH	Ew	D
					Maches	
1	Gozenyu No. 1	19°C	48.1°C	7.6	4.06	
2	〃 No. 2	21	44.5	〃	1.17	
3	Otsugiyu No. 1	21	48	〃	4.50	49m
4	〃 No. 2	21	48	〃	3.87	〃
5	Shinyu	22	47.7	〃	3.41	〃
6	Matsunoya	21	46.2	〃	3.28	42
7	Matsunoya Betsukan	25	45.9	〃	2.36	49
8	Ryûsensô	25	40.6	〃	1.53	46
9	Kumamotoya	25	44	〃	1.85	36
10	Sumitaya	〃	47.4	〃	2.88	40

11	Ootsuka	25	31.5	7.6	0.72	55
12	Ishikawaya	〃	39.8	〃	1.00	27
13	Hiraya	〃	42.5	〃	1.22	44
14	Arataya	23	43.5	〃	1.78	49
15	Yatsushiroya	〃	43.2	〃	1.66	46
16	Kadoshiroya	〃	45.7	〃	2.42	40
17	Taishokan	〃	46.5	〃	1.52	25
18	Shimaya	〃	43	〃	1.39	51
19	Umeya	〃	42	〃	0.41	49
20	Yanagiya Shintaku	〃	43.1	〃	0.48	67
21	Nakajima	〃	39.8	7.8	5.44	24
22	Ozawa	21	42.5	7.6	1.11	27
23	Kimparô No. 1	24	44.7	〃	0.63	91
24	〃 No. 2	23	44.6	〃	0.76	〃
25	Izumiya Honten	21	48.0	7.8	1.15	49
26	Izumiya Shiten	23	43.7	7.6	0.87	31
27	Nagasuya	19	48.0	〃	1.26	55
28	Kagamiya	23	46.5	〃	1.06	49
29	Bushiya	〃	42	7.4	0.86	49
30	Akaboshi	〃	46.1	7.8	2.07	48
31	Matsu-no-yu	〃	46.9	7.6	1.39	51
32	Nishiyu	〃	43.8	〃	0.78	55
33	Fukuzumiya	20	38.8	7.2	1.00	55
34	Hinaguya	〃	36.5	7.0	0.72	49
35	Sankaisô	23	31.5	7.4	0.89	73

Note D; Depth.
Ta; Atmospheric temperature in °C.
Ts; Temperature of spring in °C.
Ew; Emanation per litre of water in Mache's units.

2. *Consideration of the Results.*

First of all, we will consider the temperature of hot springs. In comparing the temprature of hot springs we must take into consideration the discharge amount and the pumping amount. This matter must be given due consideration because hot spring water cools as it rises. We can, however, neglest the drop in temperature when water is pumped in large amount. Therefore, I decided to consider only the hot springs where motors were used to pump water. Table 6-2 shows the values of temperature measurments with a maximum thermometer at Kagamiya which is a new spout completed in July 1952 by the owner Mr. Shimada. Actually a small holes were bored in the iron pipe

Table 6-2

Depth of Boring	Temperature of Spring
5m～7m	24.5°C～25°C
20m	42°C
31m	45.8°C
35 〃	48 〃
37 〃	48.8 〃
42 〃	50.2 〃
45 〃	50.3 〃
46 〃	50.5 〃
49 〃	50.5 〃

within the scope of 30m-49m and water is pumped from these limits. It seems most suitable to divide the hot springs into three groups according to the depth of boring. I shall consider only those in the 36m-54m depth since the number of the shallower and deeper ones are very few. Now, the drawing of iso-thermal curves are indicated in Fig. 6-3. From this curve there

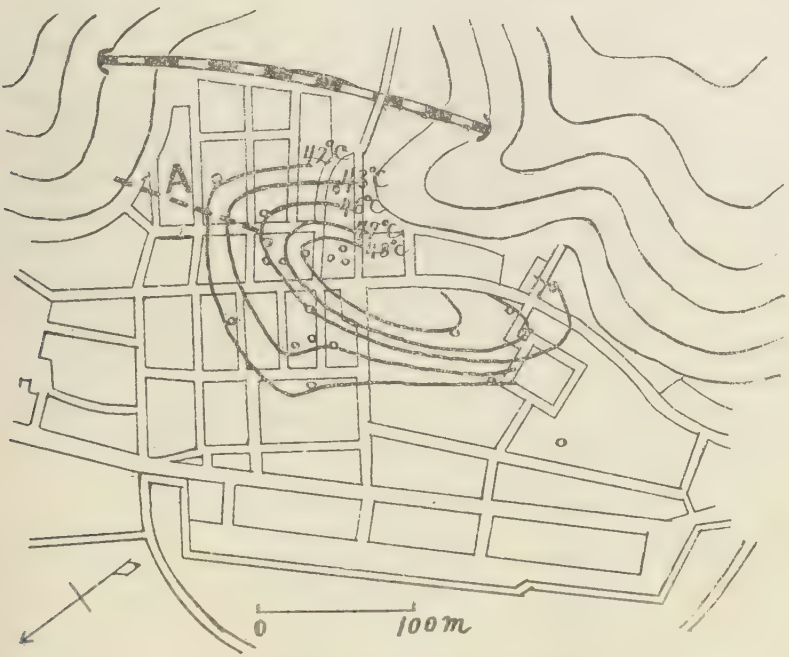

Fig 6-3 Iso-thermal Curves (36m~54m)

is a fault in the direction of line A and it may be assumed that it is from here that the hot spring water spouts. The irregularities in the iso-thermal curves as it approaches the beach is most likely due to the mixing of the ground-water. The percentage of mixing of ground-water can be deduced from the study of the chemical composition. Next, the drawing of iso-radon curves of hot springs within the boring depth of 36m-54m are as indicated in Fig. 6-4. It can be inferred that a fault exists from this curve in the direction of line B. Line A in Fig. 6-3 and line B in Fig. 6-4 practically coincide. Assuming the spouting mechanism of Hinagu Hot Springs to be thus and that such distribution was due to the

Table 6-3

Quantity of Radon	(mache' s units)	4	3	2	1
Time	(day)	1.6	2.2	3.8	
Current Velocity	(m/day)	16	15	14	

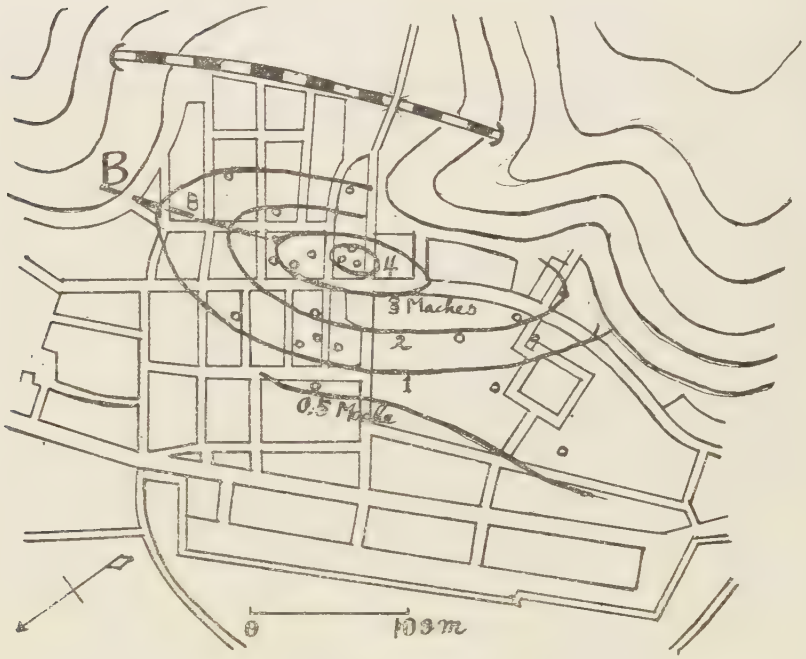

Fig 6-4 Iso-radon Curves (36m~54m)

difference in the time of passage underground untiil the time of spouting, calculations obtained are as shown in Table 6-3. The table indicates that it required 2.2 days for radon to decrease from 3 maches to 2 maches, and the current velocity of the hot spring water underground as calculated from iso-radon curves is about 15m/day.

Sectin 7. Radon Distribution at Hitoyoshi Hot Springs.

1. General Facts and Results of Measurements.

As shown in Fig. 6-1 Hitoyoshi Hot Springs are located in Hitoyoshi City, situated midway on the Kuma-gawa in southern part of Kumamoto Prefectur and may be classified as carbonated springs. There are at present hot springs spouting in about 30 places and the water is pumped in most places. The statical water head at No. 5 Kasenso is about 2 meters below the surface of the earth, about 1 meter at No. 15 Nagami, and about 1 meter above the earth at No. 17 Hitoyoshi and No. 20 Kojo. No. 16 Hayase and No. 22 Yoroin are not using pumps, but there seems to be enough volume spouted to take care of hot spring water for operation of the hotel. Both No. 25 Hayashi and No. 27 Chinryuso are about 2 meters below the surface of the earth. The depth of boring is about 150 meters at Hayashi Onsen on the west side of Hitoyoshi City and becomes deeper proceeding eastward, the deepest being at Yoshino Onsen at the depth 590 meters.

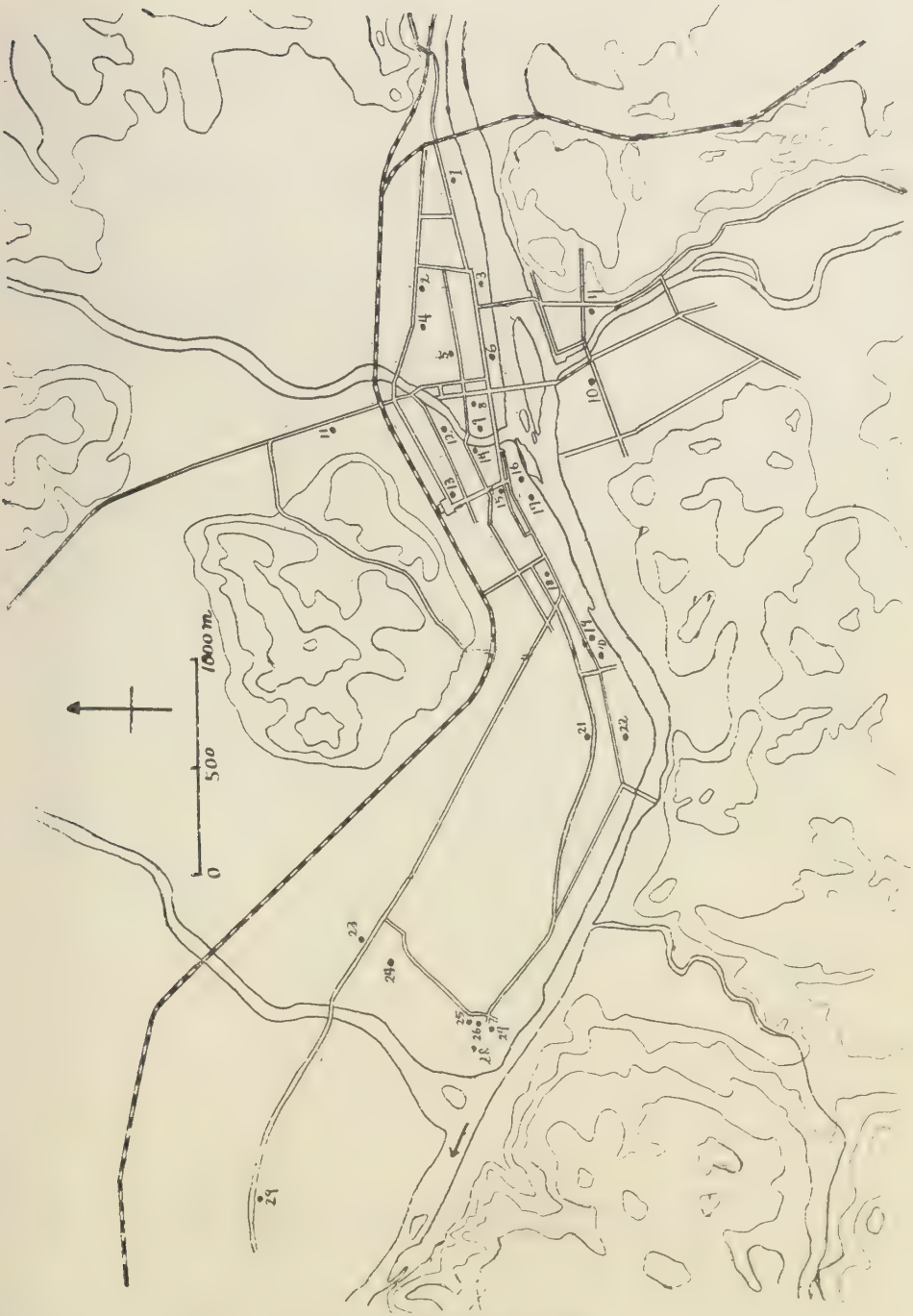

Fig 7-1 Sites of Hot Springs

The temperature is about 38°C to 48°C, with an average of 42°C. The distribution of the hot spring spouts is shown in Fig. 7-1. The method of measurement is the same as mentioned before. Measurements were taken on the spot during the period of January 5 to January 20, 1954. The results of the measurments are shown in Table 7-1. Hayashi Onsen is quite old but the other hot springs were dug, starting with Suiranro about 1910, Yoroin 1927 and spreading eastward, forming a comparatively new hot springs' town.

Table 7-1 Radon Content of Hitoyoshi Hot Springs

No.	Spring	Ta	Ts	pH	Ew	D
					Maches	
1	Myōkō	7.5°C	41.5°C	7.2	0.90	545m
2	Shōwa	10	43.5	7.8	0.60	460
3	Miuraya	7.5	46.5	7.4	1.10	460
4	Hirayama	10	45.5	7.8	0.71	545
5	Kasensō	10	45	7.6	0.90	473
6	Nabeya	9	43.8	7.6	0.68	448
7	Motoyu	9	42.1	7.5	1.67	491
8	Uguisu	10	43.5	7.2	1.09	430
9	Arata	15	44.5	7.8	0.90	418
10	Tsutsumi	9	38	7.1	0.66	510
11	Tsurukame	7.5	42	8.2	0.65	509
12	Chûō	6	40	—	—	418
13	Aoyagi	6	46.5	8.0	1.05	364
14	Yoshino	15	44.5	7.8	0.76	590
15	Nagami	15	45.5	7.6	1.63	419
16	Hayase	15	39.0	7.2	1.03	306
17	Hitoyoshi	10	42.5	7.6	1.84	218
18	Asahi	10	45.5	7.8	2.23	236
19	Sagara	10	43	7.8	1.76	363
20	Kojō	10	43	7.8	2.27	363
21	Mahara	10	39	8.0	1.16	181
22	Yōrōin	10	42	8.0	1.73	273
23	Meotomatsu	6	41	7.5	1.47	309
24	Nishihashi	6	41.5	7.9	1.38	306
25	Hayashi	10	42	7.8	1.09	200
26	Furō	11	47.5	7.8	1.35	157
27	Chinryusō	10	45	7.8	0.81	154
28	Suiranrō	11	48	7.8	0.88	122
29	Nakagami	6	40	7.8	1.37	133
30	Yoshio Onsen	15	37.0	7.2	0.22	0
31	Yuyama Onsen	10	37	9.63*	1.39	0

Note; Mark * indicates that measurements were made with Beckman G type pH meter

2. Consideration of the Results.

At Hitoyoshi there are a considerable discharg amount of hot spring water regardless

Fig 7-2 Iso-thermal Curves of Hot Springs.

Fig 7-3 Iso-Radon Curves

Fig7-4 Iso-thermal Curves of well.

of whether a motor-pump is used or not and as the depth of boring is deep in all cases the hot springs are not divided according to the depth as in the foregoing case. The isothermal curves are shown in Fig 7-2, which indicates certain irregularities along the lines A and B. The iso-radon curves are shown in Fig. 7-3, lines C and D suggest some underground irregularities. Also the depth, which is 200 meters at Hayashi Onsen, becomes deeper the boring situates to the eastern side, but near the line A it becomes a little shallower and then deepens again. In the vicinity of line A there are low temperature hot springs of 39°C-40°C. Fig. 7-4 is the distribution of temperature of ground-water investigated by Mr. Tsukuda of Hitoyoshi Upper Secondary School. H indicates the high temperature areas. On comparing Fig. 7-2, 7-3, line A and line C, line B and line D almost coincide, respectively. H_1 in Fig. 7-4 is practically identical with line A, H_2 swerves a little to the east. Also H_3 swerves slightly to the east of line B but it can be considered as practically coincide. Furthermore, the temperature of hot springs in the proximity of line A is low but the temperature of ground-water is high. There are no hot springs in the proximity of line B. From these facts line A and line B indicate the direction of the fault and it can be assumed that in this vicinity the ground-water mingles with the hot spring water, thus lowering the temperature of the hot spring water and raising that of the ground-water. The current velocity as obtained from iso-radon curves is approximately 40m/day. This seems a little too fast.

3. Other Hot Springs.

Yoshio Hot Springs is located in the river bed of one of the tributaries of Kuma-gawa as shown in Fig. 6-1. The radon content is very small.

Yuyama Hot Springs is locatd at the foothill of Ichifusa Mountain on the borderline of Miyazaki and Kumamoto prefectures. The pH was very high. A value of 9.63 was indicated by measuring with Beckman G type pH meter. The base rock is fine grained biotite-hornblende grano-diorite. The maximum of 9.71 pH value, by the same pH meter was obtained at Yamaga Hot Springs in the north. The base rock at Yamaga is hornblende-biotite grano-diorite.

Acknowledgment.

I wish to express my thanks and appreciation for the valuable instruction given me by Prof. Namba. I also thank Prof. Ochiai who encouraged and accorded me many conveniences. I wish to express my appreciation for the various assistance rendered me by Principal Uemura and others at the spot.

A part of the expence of this investigatiion was defrayed from the Scientific Research Grant from the Ministry of Education, to whom I extend my thanks.

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ON DIFFERENTIAL EQUATIONS IN LOCALLY CONVEX SPACES OF SOME TYPES.

Mituo INABA

(Received January 10, 1955)

1. The widest generalization of the theorems on existence of solutions of the differential equation:

$$\frac{dx}{dt} = f(x, t) \qquad (1)$$

where x denotes a point of the n-dimensional vector space R^n and t a real number, to the case of an infinite-dimensional vector space E, seems not to have ever been given. It has already been classical that Piacrd's existence theorem, i.e. the existence theorem under Lipschitz condition holds true in any Banach space. But Peano's existence theorem, i.e. the existence theorem under the condition of continuity alone of the functon $f(x, t)$ cannot always be valid. In fact, an example to the contrary was given by J. Diudonné [1][(1)].

On the other hand it was proved by A. Tychonoff [2] that Peano's existence theorem holds true in case E is a product space $\prod_{\lambda\in\Lambda} R^1_\lambda$ of one-dimensional spaces R^1_λ with product topology. Therefore, in our case of generalization of n-dimensional space to infinite-dimensional ones, it might be more natural to consider E as a product space than as a Banach space.

Now, as is easily seen, Peano's existence theorem bases its validity on that of Arzelà's theorem, whose proof is essentially based on the two properties, i.e. the completeness of the space E and the property, which shall be called here temporarily *property* (M) and is stated as follows: "every bounded subset of the space is relatively compact." These two properties are preserved on the procedure of product of spaces [3, Chap. I, 10, Th. 2, Cor. and Chap. II, 5, Prop. 4]. A Banach space provided with these two properties is no other than finite-dimensional, which is trivial. Among infinite-dimensional vector spaces with these two properties, most particular is a space (M), i.e. a locally convex, metrisable, and complete space, with these two properties [4, p. 79]. This space has almost the same character as a finite-dimensional space R^n. Hence it will be natural to expect the validity of Peano's existence theorem in a space (M), and moreover, after Tyconoff's procedure [2], in a product space of spaces (M) with product topology, which will be verified in the following,

2. Let E be a space (M). Since E is metrisable, its topology is defined by a denombrable sequence of semi-norms $\{p_n\}$ [5, Chap. II, 5, Prop. 6]. Let $x(t)$ denote a function

(1) Numbers in brackets refer to the bibilography at the end of the paper.

on the closed bounded interval $I=[a, b]$ to E.

Lemma 1. *If the sequence of fuuctions* $\{x^m(t)\}$ *is equicontinuous in the interval I and is convergent on a subset* M *dense in the interval I, then it converges uniformly in the whole interval I.*

The proof is analogous to that of the one-dimensional space [6, p. 61, Hilfsatz 2], with slight modifications on uniform continuity as follows: for an arbitrary positive number ε and an arbitrary set of semi-norms $\{p_{n_1}, p_{n_2}, \cdots\cdots, p_{n_k}\}$, there exists a positive integer N, such that the inequalities

$$m, m' \geqq N$$

imply those

$$p_{n_i}(x^m(t) - x^{m'}(t)) < \varepsilon, \qquad (i = 1, 2, \cdots\cdots, k)$$

for every $t \in I$.

Lemma 2. *If functions of the sequeuce* $\{x^m(t)\}$ *are equicontinuous in the interval I and bounded at every point of the interval I, then there exisits such a subsequence* $\{x^{m'}(t)\}$ *as converges uniformly in the whole interual I.*

The proof follows, by virtue of the property (M), from Lemma 1, analogously to the case of the one-dimensional space [6, p. 60, Hilfsatz 1].

We denote the set of all functions $x(t)$ on I to E, continuous in the interval I by $\boldsymbol{C}(I; E)$ or simply $\boldsymbol{C}$, and each function $x(t)$ considered as element of $\boldsymbol{C}$ by $\boldsymbol{x}$. Let $\boldsymbol{x}+\boldsymbol{y}$ denote the sum function $x(t)+y(t)$ and $\alpha\boldsymbol{x}$, where α denotes a real (or complex) number, the product function $\alpha x(t)$ multiplied by α, and then the set $\boldsymbol{C}$ will be a vector space. Moreover, if we define $q_n(\boldsymbol{x})$ as follows:

$$q_n(\boldsymbol{x}) = \sup_{t \in I} p_n(x(t)), \qquad \cdots\cdots (2)$$

then each q_n is, as is easily seen, semi-norm of the vector space $\boldsymbol{C}$. Thus the vector space $\boldsymbol{C}$ will be topologized by a denombrable sequence of semi-norms $\{q_n; n = 1, 2, \cdots\cdots\}$ and then shall be denote by $\boldsymbol{C}(I, E; q_n, n = 1, 2, \cdots\cdots)$ or simply by $\boldsymbol{C}(I, E; q_n)$. The completeness of the space E implies that of the space $\boldsymbol{C}(I, E; q_n)$.

Since the space $\boldsymbol{C}(I, E; q_n)$ is metrisable, compactness and sequential compactness are coincident, and threfore we shall have the following fundamental

Lemma 3. [*Arzelà*] *A subst* F *of the space* $\boldsymbol{C}(I, E; q_n)$, *consisting of functions* $x(t)$ *equicontinuous in the interval* I *and uniformly bounded there, is relatively compact.*

3. Let E be a space (M), and $f(x, t)$ a function on $E \times I$ to E, where $I=[t_0, t_0 + \tau]$.

Theorem. Let the function $f(x, t)$ *be continuous and bounded in* $E \times I$, *i.e. for every* n *let there exists a positive constant* M_n *such that*

$$p_n(f(x, t)) \leqq M_n .$$

Then there exists at least one solution of the differential equation (1):

$$\frac{dx}{dt} = f(x, t)$$

which satisfies the initial condition: $x = x^0$ *for* $t = t_0$ *and is defined in the interval* I.

Proof. The problem can be reduced to the existence of solutions of the integral equation:

$$x(t) = x^0 + \int_{t_0}^{t} f(x(t), t)dt, \qquad x(t) \in \boldsymbol{C}(I, E; q_n) \quad \cdots\cdots\cdots\cdots (3)$$

Now, let $\boldsymbol{X}$ the set of all functions $\boldsymbol{C}(I, E; q_n)$ which satisfy the following conditions:

$(a)_n$ $\quad p_n(x(t) - x^0) \leqq \tau M,$

$(b)_n$ $\quad \limsup_{h \to 0} p_n\left(\frac{x(t+h)-x(t)}{h}\right) \leqq M_n .$

Then the set $\boldsymbol{X}$ will be easily found to satisfy the conditions of Lemma 3, and therefore it is relatively compact. Since it is closed, it is compact, and obviously convex.

The relation

$$y(t) = x^0 + \int_{t_0}^{t} f(x(t), t)\, dt \cdots\cdots\cdots\cdots (4)$$

will define a transformation Φ:

$$y = \Phi(x)$$

which transforms the set $\boldsymbol{X}$ into $\boldsymbol{X}$:

$$\Phi(\boldsymbol{X}) \subset \boldsymbol{X}.$$

Indeed,

$$p_n(y(t) - x^0) = p_n\left(\int_{t_0}^{t} f(x(t), t)\, dt\right) \leqq \int_{t_0}^{t} p_n(f(x(t), t))\, dt$$

$$\leqq \tau M_n ,$$

and

$$\limsup_{h \to 0} p_n\left(\frac{y(t+h)-y(t)}{h}\right) = \limsup_{h \to 0} p_n\left(\frac{1}{h} \int_{t}^{t+h} f(x(t), t)\, dt\right)$$

$$= \limsup_{h \to 0} \frac{1}{|h|} \left(\int_t^{t+h} p_n (f(t), t)\, dt \right)$$

$$\leqq \limsup_{h \to 0} \frac{1}{|h|} \cdot |h| \cdot M_n = M_n$$

The transformation is continuous. Indeed, let

$$y = \Phi(\bar{x})$$

and then

$$p_n (y(t) - \bar{y}(t)) = p_n \left(\int_t^t \{ f(x(t), t) - f(\bar{x}(t), t) \}\, dt \right)$$

$$\leqq \int_{t_0}^t p_n \{ f(x(t), t) - f(\bar{x}(t), t) \}\, dt$$

Now let ε, ε' be arbitrarily given positive numberss, such that $\varepsilon' < \varepsilon$. Because of the continuity of the function $f(x, t)$, there exist a finite set of positive integers $\{m_1^n, m_2^n, \cdots, m_{l_n}^n\}$ and that of positive numbers $\{\delta_1^n, \delta_2^n, \cdots\cdots, \delta_{l_n}^n\}$ such that the inequalities:

$$p_{m_j^n} (x(t) - \bar{x}(t)) < \delta_j^n \qquad (j = 1, 2, \cdots\cdots, l_n)$$

imply those

$$p_n \{ f(x(t), t) - f(\bar{x}(t), t) \} < \frac{\varepsilon'}{2}. \qquad \cdots\cdots (5)$$

Hence much more the inequalities

$$q_{m_j^n} (x - \bar{x}) < \delta_j^n \qquad (j = 1, 2, \cdots\cdots, l_n)$$

imply (5), accordingly

$$p_n (y(t) - \bar{y}(t)) < \varepsilon',$$

therefore

$$q_n (y - y) \leqq \varepsilon' < \varepsilon.$$

According to the above argument, for any pair of a finite set of positive integers $\{n_1, n_2, \cdots\cdots, n_k\}$ and that of positive numbers $\{\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_k\}$, there exist a finite set of positive integers $m_j^{n_i} \begin{pmatrix} i = 1, 2, \cdots\cdots, k \\ j = 1, 2, \cdots\cdots, l_i \end{pmatrix}$ and that of positive numbers $\delta_j^{n_i}$ $\begin{pmatrix} i = 1, 2, \cdots\cdots, k \\ j = 1, 2, \cdots\cdots, l_i \end{pmatrix}$ such that the inequalities

$$q_{m_j^{n_i}} (x - \bar{x}) < \delta_j^{n_i}$$

imply those

$$q_{n_i}(y - \bar{y}) < \varepsilon_i ,$$

which shows the continuity of the transformation $y = \Phi(x)$. Hence from Tychonoff's fixed points theorem follows that the transformation $y = \Phi(x)$ of the compact convex subset X into itself has at least one fixed point $x \in X$: $x = \Phi(x)$, which proves the theorem.

4. Let E_λ be a space (M) for each index $\lambda \in \Lambda$, and its defining sequence of semi-norms $\{p_{\lambda,1}, p_{\lambda,2}, \cdots\cdots, p_{\lambda,n}, \cdots\cdots\}$. The product space $E = \prod_{\lambda \in \Lambda} E_\lambda$ with product topology is complete, since so is each E_λ $(\lambda \in \Lambda)$, and is provided with the property (M) (but not necessarily a space (M) except in case the set of index Λ is at most denombrable). Each point x of the space E can be represented by

$$x = (\cdots\cdots, x_\lambda, \cdots\cdots), \qquad x_\lambda \in E_\lambda, \qquad \lambda \in \Lambda.$$

Let $f(x, t)$ denote a function on the product space $E \times I$ to E and $f_\lambda(x, t)$ its projection on E_λ, and then the function $f(x, t)$ is represented as follows:

$$f(x, t) = (\cdots\cdots, f_\lambda(x, t), \cdots\cdots), \qquad \lambda \in \Lambda.$$

The boundedness of the function $f(x, t)$ is represented as follows: for every λ, n there exists a positive number $M_{\lambda,n}$ such that

$$p_{\lambda,n}(f(x, t)) \leqq M_{\lambda,n} .$$

The theorem will be found to hold true in the product space E by some modifications of Tychonoff's procedure [2] as follows. Let C, X, p_n, M_n corresponding to each E_λ be denoted by C_λ, X_λ, $p_{\lambda,n}$, $M_{\lambda,n}$, and then X_λ is compact and convex. Each point x of C corresponding to product space $E = \prod_{\lambda \in \Lambda} E_\lambda$ is represented by

$$x = (\cdots\cdots, x_\lambda, \cdots\cdots), \qquad x_\lambda \in C_\lambda ,$$

and C itself is a product space of C_λ $(\lambda \in \Lambda)$ with product topology [3, Chap. I, 8, Prop. 2, Cor. 1]:

$$C = \prod_{\lambda \in \Lambda} C_\lambda .$$

The space C is complete as product space of complete spaces. Let X denote the product space $\prod X_\lambda$ of subsets X_λ $(\lambda \in \Lambda)$, and then X is also compact and convex as product of such ones. On the other hand X is considered as a set of point x of C satisfying the following condition:

$$(a)_{\lambda,n} \qquad p_{\lambda,n}(x(t) - x^0) \leqq M_{\lambda,n} ,$$

$$(b)_{\lambda,n} \qquad \limsup_{h \to 0} p_{\lambda,n}\left(\frac{x(t+h) - x(t)}{h}\right) \leqq M_{\lambda,n} .$$

Since the topology of the product space $E=\prod_{\lambda\in\Lambda} E_\lambda$ is defined by the family of seminorms $\{p_{\lambda,n};\ \lambda\in\Lambda,\ n=1,2,\cdots\cdots\}$, the remaining process is quite analogous to that of Tychonoff [2].

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REMARKS ON DUALITY IN LINEAR SPACES

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(Recieved January 10, 1955)

1. Introduction. Given any locally convex topological real or complex linear (Hausdorff) space E, a topology t for its conjugate space E^* is called reflexive if E is algebraically isomorphic to its second conjugate space E^{*t*} under the "*natural*" *mapping* (cf. [1] [1]). It is well known that if the closed convexification of a compact subset in E is again compact, the k-topology for E^* (i.e. the topology for E^* according to which convergence of functionals means uniform convergence on the compact subsets of E) is reflexive. In this paper we will note that this condition is also *necessary* for reflexivity of the k-topology for E^*.
Therefore, it becomes clear that the following conditions are equivalent:

(1) The k-topology for E^* is reflexive.

(2) The closed convexification of a compact subset in E is again compact.

(3) The k-topology for E^* is equivalent to the c-topology i.e. the topology of uniform convergence on the convex, compact subsets of E.

(4) The k-topology for E^* is weaker than the κ-topology i.e. the topology of uniform convergence on the convex, weakly compact subsets of E.

M. F. Smith considered the following question in his foregoing paper (cf. [2]). "*For what class of topological groups does the Pontrjagin duality theorem hold ?*" He answered this question in part, by showing that in reflexive real linear spaces and real Banach spaces the Pontrjagin Duality Theorem is valid. By applying our result, we shall answer the above question completely in the case that the considered topological group is a locally convex topological real linear (Hausdorff) space. Our answer is stated as follows. Let E be a locally convex topological real linear (Hausdorff) space. In order that the Pontrjagin Duality Theorem should be valid in E as a topological group, the following conditions are necessary and sufficient:

(1) The closed convexification of a compact subset in E is again compact.

(2) Any k-compact subset of the conjugate space E^* is equicontinuous.

2. Preliminaries.

Let E be a locally convex topological real or complex (Hausdorff) space, E^* its conjugate space, i.e. the space of continuous linear functionals on E. The zero vector in E is denoted by θ, the zero functional on E by θ^*. Let E^{*t} be the space E with the topology t. When we wish to talk of a notion in the t-topology, we shall refer to t-notion —— e.g.

1) Numbers in brackets refer to the bibliography at the end of the paper.

t-compactness, t-neighborhood. A topology t for E^* is called *reflexive* if E is algebraically isomorphic to E^{*t*} under the "*natural*" *mapping* N defined by

$$N(x) = X \text{ if and only if } f(x) = X(f) \text{ for all } f \in E^*. \quad (x \in E,\ X \in E^{**}.)$$

The polar in E^* of a set K in E, denoted by K°, is the set of all f in E^* such that, if x is in K, then $|f(x)| \leqq 1$. *The polar in* E of a set K in E^* is defined as the set of all x in E such that, if f is in K, then $|f(x)| \leqq 1$. Furthrmore, a set K such that, if $\lambda + \mu = 1$, and if $\lambda, \mu \geqq 0$, then $\lambda K + \mu K \subset K$, is convex, a set K such that, if $|\lambda| = 1$, then $\lambda K \subset K$, is *circled*. The smallest convex enveloped of a set K is called the *convexification* of the set K, and denoted by K_{conv}. The closure of a set K is denoted by K^b.

We use the notions of the k-, p-, c-, *weak*, κ-*topologies* and the topology which we call k^*-*topology*. The topology for E^* of uniform convergence on compact subsets of E is called k-*topology*, and k-neighborhoods of 0^* are the sets K°, where K is a compact subset of E. The topology (for E^*) of simple convergence, is called p-*topology*, and p-neighborhoods of 0^* are the sets K°, where K is a finite set of E. Furthermore, we call c-*topology* the topology for E^* of uniform convergence on convex, compact subsets of E, and c- neighborhoods of 0^* are the set K°, where K is a convex compact subset of E. The *weak topology* for E has neighborhoods of 0 of the form K°, where K° is the polar in E of a set K in E^*. The topology for E^* of uniform convergence on convex, weakly compact subsets of E, is called κ-*topology*, and has neighborhoods of 0^* of the form K°, where K is a convex, weakly compact subset of E. The k^*-*topology* for E has neighborhoods of 0 of the form K° where K is the polar in E of a k-compact set in E^*. This k^*-topology is stronger than original topology for E.

Next, assume that E is a locally convex topological real linear (Hausdorff) space, and that E is the character group of E as a topological group. The Pontrjagin Duality Theorem, known to be true for locally compact groups, asserts that the given group E and the character group E of its character group E are isomorphic under the "*natural*" *mapping* M defined by

$$M(x) = \Phi \text{ if and only if } \varphi(x) = \Psi(\varphi) \text{ for all } \varphi \in \bar{E}. \quad (x \in E,\ \Phi \in E.)$$

We shall answer the following question completely in section 4: "*For what class of linear spaces as topological groups does the Pontrjagin theorem hold?*" We shall denote the set of all f satisfying the condition X by $\{f;\ X\}$.

3. Reflexivity of k-topology.

Let E be a locally convex topological real or complex linear (Hausdorff) space, E^* its conjugate space. We prove at first the following

LEMMA. *If* A *is a* k-*neigoborhood in* E^*, *then* k^*-*topology is equivalent with weak topology*

on A°, where A° is the polar in E.

Proof. To prove this lemma, it is sufficient to show that, corresponding to each k^*-neighborhood $N^{k^*}(x_0)=x_0+C^\circ$ of a point $x_0 \in A$, where C is a k-compact set in E^*, a weak neighborhood $N^w(x_0)$ of a point x_0 can be found, such that

$$N^w(x_0) \cap A^\circ \subset N^{k^*}(x_0) \cap A^\circ.$$

Now, let $N^k(f)=f+(1/4)A$ $(f \in E^*)$, and let $N_f^w(\theta)=\{4f\}^\circ$. Then, if $g \in K^k(f)$, $x \in N_f^w(\theta) \cap A^\circ$, then $4(g-f) \in A$, $x \in A^\circ$, and $x \in \{4f\}^\circ$. Therefore, we have two inequalities:

$$|g(x)-f(x)| \leqq 1/4, \qquad |f(x)| \leqq 1/4.$$

Hence, $|g(x)| \leqq |g(x)-f(x)|+|f(x)| \leqq 1/2$.

That is, corresponding to each $f \in C$, there exists a k-neighborchood $N^k(f)$ and a weak neighborhood $N_f^w(\theta)$, such that $|g(x)| \leqq 1/2$ for all $g \in N^k(f)$ and $x \in N_f^w(\theta) \cap A^\circ$. Let $f_1, \ldots, f_n$ be a finite number of points in C such that $\bigcup_{\alpha=1}^n N^k(f_\alpha)$ covers C, and let $N^w(x_0)=x_0+\bigcup_{\alpha=1}^n 2N_{f_\alpha}^w(\theta)$. Now, assume that $x \in N^w(x_0) \cap A^\circ$. If $f \in C$, then $f \in N^k(f_\alpha)$ for some α, and $1/2\ x-1/2x_0 \in N_{f_\alpha}^w(\theta)$. Since A° is a convex circled set, we have the relation $1/2\ x-1/2\ x_0 \in A^\circ$. Therefore, $1/2\ x-1/2\ x_0 \in N_{f_\alpha}^w(\theta) \cap A^\circ$. Hence $|f(1/2\ x-1/2x_0)| \leqq 1/2$. That is, $|f(x-x_0)| \leqq 1$. Therefore, $x-x_0 \in C^\circ$. That is, $x \in x_0+C^\circ=N^{k^*}(x_0)$. Hence, $N^w(x) \cap A^\circ \subset N^{k^*}(x_0) \cap A^\circ$.

Next, using the above lemma, we prove our main

THEOREM 1. *Let E be a locally convex topological real or complex linear (Hausdorff) space, E^* its conjugate space. If the k-topology for E^* is reflexive, then the closed convexification of a compact subset in E is again compact.*

Proof. Assume that the k-topology for E^* is reflexive, and let K be a compact set in E. Then the polar $A=K^\circ$ in E^* is a k-neighborhood. Since the polar $A^\circ=K^{\circ\circ}$ in E^{**} is p-compact, and the k-topology for E^* is reflexive, the polar $A^{\circ\circ}=K^{\circ\circ\circ}$ in E is weakly compact. Therefore, by the above lemma, $A^{\circ\circ}=K^{\circ\circ\circ}$ is k^*-compact. Hence, $K^{\circ\circ\circ}$ is a convex, circled, compact set in E. Since $K_{conv}^b \subset K^{\circ\circ}$, we see that K_{conv}^b is compact.

From the above theorem 1, we have the following

COROLLARY. *Let E be a locally convex topological real or complex linear (Hausdorff) space, E^* its conjugate space. The following four conditions are equivalent:*

(1) *The k-topology for E^* is reflexive.*

(2) *The closed convexificaiton of a compact subset in E is again compact.*

(3) *The k-topology for E^* is equivalent to the c-topology.*

(4) *The k-topology for E^* is weaker than the κ-topology.*

4. The Pontriagin Duality Theorem in linear space.

From theorem 1, we have the following

THEOREM 2. *Let E be a locally convex topological real linear (Hausdorff) space. In order that the Pontrjagin Duality Theorem should be valid in E as a topological group, the following conditions are necessary and sufficient:*

(1) *The closed convexification of a compact subset in E is again compact.*

(2) *Any k-compact subset of the conjugate space E^{*k} is equicontinuous.*

Proof. The character group E of E as a topological group and E^{*k} are algebraically and topologically equivalent as topological groups under the mapping T of E^{*k} into $\bar{E}$ defined by

$Tf = \chi$ where $\chi(x) = \exp[if(x)]$ for all $x \in E$.

(cf. [2]). Indeed, by lemma 1 in [2], E^{*k} and E are algebraically isomorphic groups. It is clear that T is continuous. For a open k-neighborhood $(K, \varepsilon) = \{f \in E^*;\ |f(x)| < \varepsilon, x \in K,$ K compact, $K \subset E.\}$ of θ^* in E^*, let $K_1 = \bigcup_{n=-\infty}^{\infty} \lambda_n K$, where $\lambda_n = (2n\pi - \varepsilon)/(2n\pi + \varepsilon)$ $(n \neq 0)$, $\lambda_0 = 1$, and let $\varepsilon_1 < \varepsilon$. Then, we see that

$$\{f \in E^*;\ |e^{if(x)} - 1| < \varepsilon,\ x \in K_1\} \subset (K, \varepsilon),$$

and K_1 is a compact set in E. Therefore the inverse mapping T^{-1} is continuous.

Therefore, to prove this theorem 2, it is sufficient to show that, in order that E^{*k*k} and E should be algebraically and topologically equivalent, two conditions (1), (2) of this theorem 2 are necessary and sufficient. By theorem 1, we see that the condition (1) is necessary and sufficient for algebraical equivalence of E^{*k*k} and E. We can show easily that the condition (2) is necessary and sufficient for topological equivalence of E^{*k*k} and E.

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SOME STUDIES ON VOLCANO ASO AND KUJIU (PART 12). ANNUAL VARIATION OF INTERNAL ENERGY OF VOLCANO KILAUEA.

Munetosi NAMBA.

(1954 Dec. 10 th.)

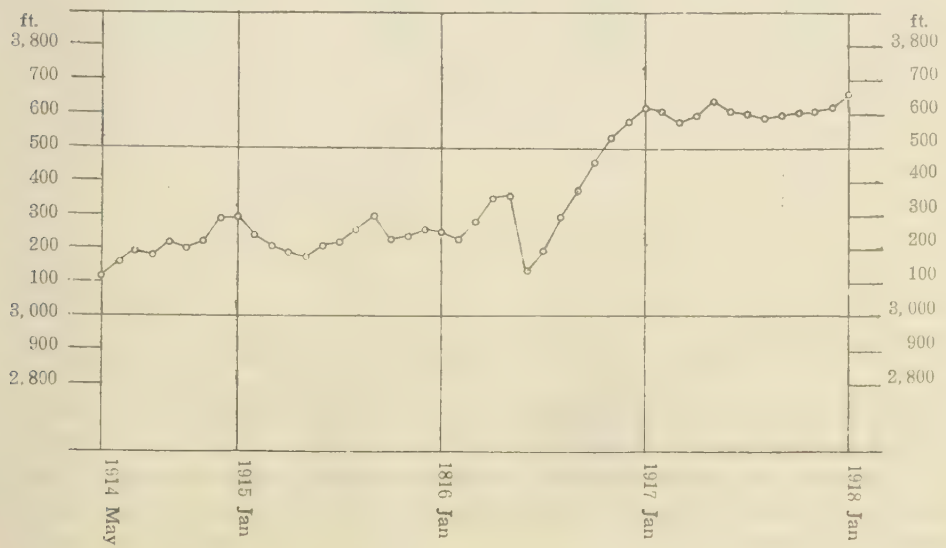
abstract.

Iuvestigation has brought a new fact- -that annual variation of lava head in Volcano Kilauea corresponds to that of the precipitation and atmospheric pressure, and semi-annual variation corresponds to the max. min. range of atmospheric pressure- -to light.

1. Foreword.

On taking the number of times of explosion for a reference scale showing the rate of intensity in energy of volcanic activity, the approximate character in many instances of annual variation of volcanic activity has come to light. (1)
And, according to quantity of spouted sulphur from the crater, the state of affairs in volcanic activity of Volcano Kujiu were also studied. (2)

In these cases, there is a tendency on rising atmospheric pressure to accerate explosion, and precipitation to restrain it.
On the contrary, scientific researches of internal energy of a volcano has never been made;



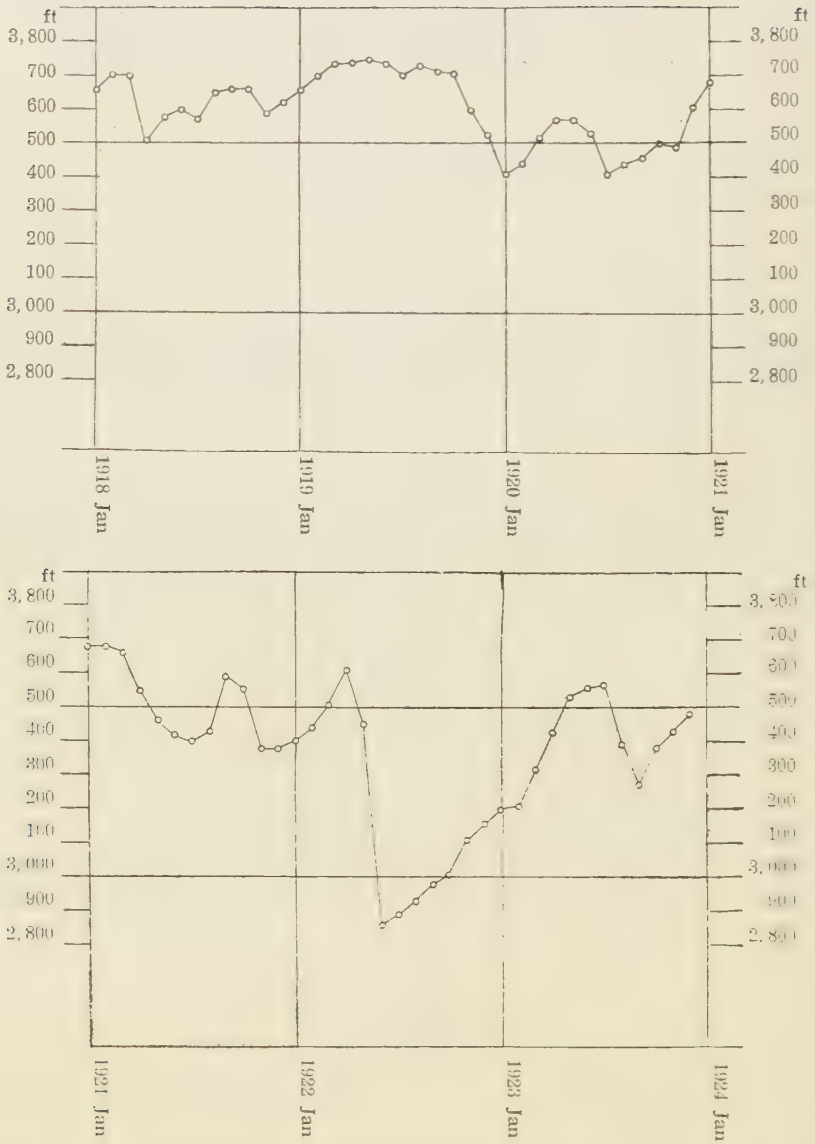


Fig 1. Lava head above the sea level in Volcano Kilauea observed by Prof. Dr. Jaggar.

however, lava head increase and decrease in the crater of Volcano Kilauea have been measured minutely. (3)

Then this writer based his studies of annual variation of internal energy of Volcano Kilauea upon data that quantity of internal energy of a volcano is the cause of lava head increase and decrease.

2. Annual Variation of Lava Head in Volcano Kilauea.

Prof. Dr. Jaggar made observation level of lava head above the Sea in Halemaumau of Volcano Kilauea for 10 years from 1914 to 1923. (4) The numerical values out of his graph and the blanks filled with interpolation, the parethesized values, are showed in Table 1.

Table 1. Lava Head above the sea in Kilauea and Precipitation in Honolulu.

year, month	lava head	precipitation	year, month	lava head	precipitation	year, month	lava head	precipitation	year, month	lava head	precipitation
	ft	mm									
1914, 1		60	1916, 7	3198	44	1919, 1	3654	38	1921, 7	3395	20
2		61	8	3297	27	2	3703	7	8	3432	23
3		127	9	3375	15	3	3740	32	9	3587	28
4		52	10	3463	31	4	3739	10	10	3550	110
5	3116	23	11	(3530)	33	5	3747	17	11	3376	11
6	3163	11	12	(3580)	141	6	3736	22	12	3380	155
7	3188	10	1917, 1	3616	212	7	3703	21	1922, 1	3401	162
8	3177	9	2	3614	53	8	3727	15	2	3436	20
9	3220	55	3	3579	403	9	3717	30	3	3509	90
10	3203	30	4	3604	41	10	3707	124	4	3612	9
11	3218	37	5	3642	42	11	3596	32	5	3451	23
12	3286	111	6	3612	14	12	3534	23	6	2856	10
1915, 1	3286	9	7	3599	14	1920, 1	3413	99	7	2887	15
2	3244	21	8	3588	28	2	3443	18	8	2931	19
3	3208	11	9	3603	18	3	3519	95	9	2979	38
4	3185	49	10	3613	37	4	3572	23	10	3054	82
5	3176	5	11	3610	78	5	3570	32	11	3113	26
6	3208	22	12	3622	117	6	3525	9	12	3157	17
7	3224	51	1918, 1	3664	150	7	3410	10	1923, 1	3195	287
8	3262	5	2	3696	81	8	3441	19	2	3206	95
9	3301	39	3	3700	122	9	3456	8	3	3315	161
10	3226	37	4	3513	174	10	3498	22	4	3429	86
11	3239	271	5	3577	10	11	3494	90	5	3531	9
12	3256	233	6	3598	22	12	3607	222	6	3561	4
1916, 1	3251	466	7	3567	16	1921, 1	3683	254	7	3569	7
2	3227	82	8	3647	69	2	3675	29	8	3389	9
3	3275	172	9	3663	11	3	3663	10	9	3270	21
4	3354	48	10	3656	36	4	3545	18	10	3377	56
5	3407	49	11	3590	135	5	3459	9	11	3434	10
6	3135	34	12	3617	126	6	3421	8	12	3483	203

The result of analysis:

$$\overset{ft}{3439.78} + \overset{ft}{51.05}\ cos\ (\theta - 39) + 27.55\ cos\ (2\theta - 165) + 13.72\ cos\ (3\theta - 315) + \cdots\cdots$$

The annual term is largest and there is no neglecting the semi-annual term.

While, on the contraary, the amount of precipitation for every month in Honolulu:

$$\overset{m.m.}{64.6} + \overset{m.m.}{53.77}\ cos\ (\theta - 12) + \cdots$$

Jadging from the data, lava head must increased at the rate of 51.05/53.8 ft/mm, that is, 0.95 ft. /1 m. m. about a month after precipitation. On this, adjusting the effect of the amount of precipitation, the annual term was derived as 16.94 ft. Cos (θ-106.7).

And atmospheric pressure in Honolulu:

$$\overset{m.m.\,Hg}{761.63} + 0.65\ cos\ (\theta - 63) + \cdots\cdots$$

Then, variation of atmospheric pressure exerts an influence on lava head increase and decrease at the rate of 16·94/0.65 ft./m.m. Hg, that is, 26 ft./1 m.m. Hg about a month after precipitation.

Synthetically judging from the above two factors, it is clear that Volcano Kilauea is in max. internal energy in Spring and min. internal energy in Autumn.

Therfore, on annual variation of internal energy, it is quite natural that Volcano Kilauea is apt to become active at Max. internal energy.

3. Semi-annual Variation of Lava Head in Volcano Kilauea.

This writer has analysed the number of times of eruption of a volcano in Japan per month, appeared in observations of the Earthquake Investigation Commitee at page 40 of its Nihon Kazan Shi.

About all volcanoes in Japan we have:-

$$52.9 + 14.2\ cos\ (\theta - 75) + 4.6\ cos\ cos\ (2\theta - 40) + \cdots\cdots \text{〔the Number of times〕}$$

About Volcano Aso we have:-

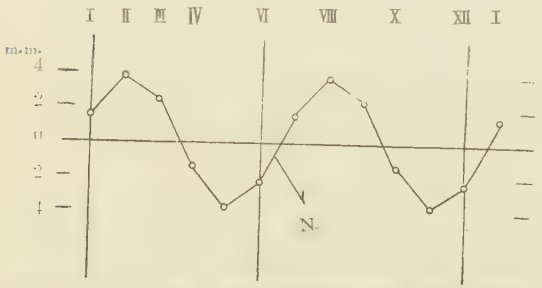
$$6.7 + 1.77\ cos\ (\theta - 74) + 1.94\ cos\ (2\theta - 12) + \cdots\cdots \text{〔the Number of times〕}$$

On this, semi-annual variation in the unmber of times of eruption of a volcano is fairly predominate, let alone its annual term. And, about spouted sulphur, it has become known that semi- annual term of the spouted sulphur from Volcano Kujiu is fairly predominant, too. (4) The facts are worthy of note. Further, there is another fact still more important. Taking the effect of annual variation of precipitation and atmospheric pressure off annual variation of lava head in Volcano Kilauea, the difference:

$$\overset{ft}{12.92}\ cos\ (\theta - 118) + 38.02\ cos\ (2\theta - 186) + \cdots\cdots$$

Fig, 2. Max. Min. Rang of Atmospheric Pressure at KUMAMOTO.

$\overset{m.m.\,Hg}{6.4}\ \cos\ (\theta - 229) + 3.8\ \cos\ (2\theta - 67) + \cdots \pm 2.2$



The corresponding semi-annual variation of Spouted suephur is showed in (1) Page 55, Fig. 1.

and, in this case, semi- annual variation is fairly predominant, too.

The semi-annual term of volcanic activity of Volcano Aso and kujiu corresponds to that of the max. min. range in atmospheric pressure. (Fig. 2) Gathering from these data, the semi-annual term of lava head in Volcano Kilauea also corresponds to that of the max. min. range in atmospheric pressure; however, it is to be regretted that this writer can not demonstr-

ate the fact in quantity, because the data of the max. min. range in atmospheric pressure in Volcano Kilauea is not in his possession.

4. Conclusion.

On studying Prof. Dr. Jaggar's Observations of lava head in Volcano Kilauea for 10 years from 1914 to 1923, the new fact... that annual variation of internal energy of Volcano Kilauea corresponds to that of the amount of precipitation and atmospheric pressure and its semi-annual variation corresponds to the max. min. range of atmospheric pressure... has come to light.

To conclude, this writer should like to express his thanks to mr. T. Asoda for his endeavours in analysis. (Dec. 10 th. 1954)

Litteratures

(1) M. NAMBA:- Effect of atmospheric pressure upon a volcanic explosion: Chikiu Butsuri, vol. 6, No. 3, 1942 Dec. , pp. 142-188.
M. NAMBA, T. Murota and T. Mitsui:-
Some studies on volcano Aso and Kujiu (part 1)-
Kumamoto Journal of Science, Series A, Vol. 1, No. 1, Oct. 1952, pp. 53-65.
(part 2):-
On the periodicity of two active Volcanos Aso and Kujiu.
(2) Ditto: (1).
(3) Prof. Dr. Jaggar:- Origin and Developement of Craters, 1947.
(4) ditto (1) page 55, Fig. 1.

SOME STUDIES ON VOLCANO ASO AND KUJU (Part 13). EARTH CURRENT AT KHAILAR

Munetoshi NAMBA

In this paper Observed data are reported, and details are reported in the following paper.

Table 1. EW-Component in mv/100meters at Khailar, Blanks are the disturbed.

(W. E.) 1942, Jan.

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2																										
3																										
4												0.03	0.04	0.05	0.04	0.03	0.02	0.05	0.02	0.04	0.07	0.05	0.03	0.04	0.47	0.036
5	0.02	0.02	0.02	0.02	0.02	0.04		0.03	0.04	0.04	0.10	0.05	0.06	0.02	0.04	0.02	0.04	0.10	0.03	0.04	0.02	0.02	0.03	0.04	0.80	0.039
6	0.02	0.02	0.02	0.02	0.02	0.04		0.04	0.04	0.04	0.03	0.03	0.02	0.02			0.02	0.03	0.04	0.02	0.02	0.02	0.02	0.02	0.62	0.028
7	0.02	0.03	0.02	0.02	0.02	0.04	0.04	0.03	0.03	0.04	0.04	0.11	0.04	0.04	0.06	0.03	0.14	0.04	0.04	0.05	0.02	0.04	0.04	0.03	1.18	0.047
8	0.02	0.02	0.02	0.02	0.03	0.03	0.02	0.04	0.04	0.04	0.04														0.36	0.033
9											0.04							0.02	0.01	0.02	0.03	0.05	0.01	0.01	0.19	0.021
10	0.05	0.02	0.01	0.02	0.04	0.03	0.04																		0.21	0.030
11																										
12																0.06	0.03	0.02	0.02	0.02	0.03	0.03	0.02	0.02	0.25	0.028
13	0.03	0.06	0.02	0.02	0.04	0.04	0.06	0.03	0.08	0.04															0.42	0.042
14																	0.10	0.05	0.02	0.02	0.01	0.02	0.08	0.06	0.36	0.045
15	0.07	0.02	0.02	0.02	0.01	0.01	0.02	0.03	0.04	0.02	0.02	0.02	0.02	0.01	0.03	0.03	0.03	0.02	0.02	0.01	0.01	0.02	0.03	0.02	0.55	0.023
16	0.02	0.02	0.01	0.02	0.02	0.02	0.01	0.02				0.02	0.03	0.02	0.02	0.03	0.03	0.03	0.01	0.03	0.01	0.04	0.02	0.02	0.45	0.021
17	0.04	0.02	0.01	0.02	0.02	0.04	0.03	0.02	0.02	0.03	0.03	0.04	0.03	0.04	0.03	0.04	0.04	0.02	0.03	0.02	0.02	0.07	0.02	0.03	0.71	0.030
18	0.01	0.02	0.02	0.01	0.01	0.01		0.02	0.02	0.05	0.04	0.02													0.23	0.021
19																										
20																										
21																										
22																	0.03	0.04	0.02	0.07	0.07	0.04	0.02	0.04	0.33	0.041
23	0.02	0.02	0.02	0.03	0.03	0.01	0.02	0.03	0.03	0.04	0.08	0.05	0.03	0.05	0.04	0.03	0.03	0.06	0.03	0.02	0.01	0.05	0.05	0.02	0.80	0.033
24	0.02	0.06	0.02	0.03	0.06	0.08	0.03	0.03	0.04	0.06	0.06	0.04	0.03	0.03	0.04	0.03	0.03	0.03	0.04	0.02	0.03	0.02	0.01	0.02	0.86	0.036
25	0.03	0.02	0.01	0.03	0.02	0.02	0.04	0.03	0.03	0.03	0.06	0.02	0.06	0.04	0.05	0.06	0.08	0.06	0.02	0.05	0.08	0.02	0.03	0.04	0.93	0.039
26	0.01	0.02	0.02	0.01	0.02	0.03	0.01	0.02	0.06	0.04	0.07	0.03	0.05	0.02	0.02	0.09	0.06	0.02	0.02	0.03	0.02	0.03	0.02	0.02	0.74	0.031
27	0.03	0.03	0.02	0.02	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.05	0.06	0.06	0.06	0.05	0.05	0.03	0.03	0.03	0.02	0.03	0.03	0.88	0.037
28	0.04	0.03	0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.04	0.05	0.03	0.05	0.03	0.02	0.04	0.02	0.02	0.06	0.03	0.02	0.04	0.04	0.04	0.87	0.036
29	0.04	0.04	0.04	0.03	0.03	0.04	0.04	0.04	0.04	0.04	0.05	0.02	0.04	0.05	0.03	0.06	0.02	0.03	0.04	0.02	0.04	0.04	0.02	0.04	0.68	0.028
30	0.02	0.04	0.04	0.04	0.03	0.03	0.02	0.02	0.04	0.03	0.04	0.03	0.03	0.04	0.03	0.02	0.04	0.03	0.04	0.02	0.02	0.02	0.04	0.03	0.68	0.028
31	0.02	0.04	0.03	0.03	0.03	0.02	0.02	0.04	0.04	0.04	0.02			0.04	0.04	0.03	0.03	0.02	0.02	0.04	0.02	0.03	0.02	0.02	0.56	0.025
total	0.86	0.86	0.73	0.76	0.78	0.80	0.80	0.85	0.97	0.97	0.93	0.66	0.65	0.61	0.63	0.70	0.83	0.73	0.51	0.57	0.50	0.65	0.57	0.56	14.63	0.033
Mean	0.029	0.029	0.024	0.025	0.026	0.029	0.026	0.030	0.031	0.032	0.030	0.022	0.021	0.020	0.021	0.023	0.027	0.024	0.017	0.019	0.016	0.021	0.019	0.019		0.031

(E. W.) 1942, Feb.

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.02	0.06	0.02	0.02	0.03	0.08	0.08	0.08	0.02	0.02	0.04	0.04	0.02	0.08	0.02	0.03	0.06	0.06	0.02	0.02	0.01	0.03	0.02	0.03	0.91	0.038
2	0.02	0.02	0.10	0.02	0.03	0.04	0.10	0.07	0.06	0.08	0.06	0.10	0.20	0.18	0.18	0.26	0.06	0.11	0.06	0.06	0.02	0.03	0.03	0.09	1.89	0.083
3	0.04	0.03	0.03	0.03	0.02	0.06	0.04	0.06	0.06	0.12	0.18	0.06	0.10	0.05	0.10	0.02	0.06	0.04	0.03	0.02	0.01	0.02	0.01	0.02	1.21	0.050
4	0.06	0.02	0.02	0.04	0.02	0.03	0.07	0.02	0.03	0.08	0.06	0.02	0.08	0.06	0.02	0.03	0.02	0.02	0.06	0.01	0.02	0.07	0.01	0.02	0.89	0.037
5	0.02	0.01	0.02	0.02	0.03	0.02	0.02	0.02	0.03	0.06	0.02	0.08	0.04	0.01	0.03	0.02	0.03	0.04	0.02	0.02	0.02	0.06	0.02	0.02	0.68	0.028
6	0.02	0.01	0.04	0.09	0.05	0.02	0.02	0.02	0.02	0.06	0.06	0.09	0.02	0.02	0.10	0.08	0.04	0.04	0.02	0.06	0.03	0.02	0.14	0.02	1.09	0.045
7	0.02	0.04	0.03	0.04	0.04	0.02	0.04	0.06	0.11	0.04	0.04	0.09	0.06	0.06	0.14	0.06	0.06	0.04	0.04	0.02	0.04	0.06	0.03	0.02	1.20	0.050
8	0.03	0.02	0.03	0.02	0.01	0.02	0.03	0.02	0.01	0.07	0.06														0.32	0.030
9																										
10																										
11																										
12																										
13																0.01									0.01	0.010
14													0.04	0.04	0.07	0.04	0.04	0.05	0.06	0.03	0.05	0.04	0.03	0.04	0.53	0.044
15	0.05	0.04	0.03	0.04	0.05	0.03	0.03	0.05	0.06	0.05	0.03	0.04	0.06	0.04	0.06	0.05	0.05	0.07	0.03	0.04	0.04	0.04	0.05	0.03	1.06	0.044
16	0.04	0.07	0.08	0.04	0.04	0.03	0.03	0.04	0.03	0.10	0.06	0.04	0.03	0.05	0.03	0.06	0.04	0.06	0.03	0.04	0.04	0.05	0.03	0.06	1.12	0.047
17	0.04	0.03	0.02	0.02	0.03	0.04	0.04	0.10	0.04	0.04	0.06	0.04	0.08	0.05	0.06	0.06	0.06	0.04	0.03	0.03	0.02	0.03	0.02	0.03	1.01	0.042
18	0.04	0.04	0.03	0.03	0.04	0.10	0.07	0.02	0.04	0.07	0.08	0.12	0.09	0.06	0.05	0.02	0.05	0.03	0.03	0.02	0.04	0.03	0.02	0.03	1.15	0.048
19	0.04	0.03	0.03	0.02	0.03	0.03	0.02	0.03	0.03	0.02	0.03	0.03	0.02	0.04	0.08	0.16	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.02	0.82	0.034
20	0.02	0.03	0.02	0.03	0.02	0.02	0.03	0.03	0.04	0.08	0.03	0.04	0.06	0.06	0.05	0.17	0.04	0.03	0.04	0.04	0.03	0.04	0.03		0.98	0.043
21															0.03	0.04	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.03	0.25	0.025
22	0.04	0.10	0.17	0.03	0.02	0.04	0.02	0.02	0.03	0.03	0.05	0.02	0.02	0.03											0.62	0.044
23												0.03	0.02	0.05	0.03	0.03	0.04	0.04	0.02	0.02	0.02	0.03	0.03	0.05	0.41	0.032
24	0.04	0.10	0.03	0.03	0.06	0.02	0.03	0.04	0.04	0.10			0.07	0.05	0.08	0.03	0.05	0.04	0.03	0.04	0.04	0.02	0.02	0.06	1.02	0.046
25	0.02	0.02	0.02	0.01	0.02	0.02	0.02	0.06	0.06	0.08	0.04	0.08	0.12	0.04	0.04	0.06	0.04	0.04	0.04	0.01	0.02	0.01	0.02	0.06	0.95	0.040
26	0.02	0.01	0.04	0.02	0.02	0.06	0.02	0.04	0.05	0.18	0.15	0.12	0.08	0.06	0.02	0.02	0.02	0.01							0.94	0.052
27																	0.03	0.02	0.04	0.04	0.03	0.03	0.02	0.02	0.23	0.029
28	0.04	0.09	0.03	0.11	0.03	0.03	0.03	0.03	0.05	0.08	0.02	0.04	0.04	0.03	0.06	0.06	0.06	0.06	0.05	0.05	0.04	0.06	0.06	0.08	1.23	0.051
29																										
30																										
31																										
total	0.62	0.77	0.79	0.66	0.59	0.71	0.74	0.81	0.81	1.36	1.07	1.08	1.25	1.06	1.25	1.31	0.89	0.88	0.69	0.62	0.56	0.72	0.64	0.73	20.61	0.041
Mean	0.033	0.041	0.042	0.035	0.031	0.037	0.039	0.043	0.043	0.072	0.059	0.060	0.063	0.053	0.063	0.062	0.042	0.042	0.035	0.031	0.028	0.036	0.032	0.038	0.044	

(E. W.) 1942. March

day \ hour	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.08	0.04	0.04	0.04	0.04	0.06	0.08	0.26	0.20	0.18	0.20	0.21	0.30	0.14	0.16	0.32	1.13	0.90	0.43	0.30	0.43	0.13	0.06	0.31	6.04	0.252
2											0.11	0.11	0.06	0.06	0.04	0.04	0.04	0.03	0.04	0.04	0.04	0.07	0.04	0.04	0.82	0.059
3	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.04	0.03	0.02	0.04	0.06	0.04		0.02	0.02	0.03	0.04	0.02	0.04	0.04	0.02	0.03	0.04	0.80	0.038
4	0.04	0.03	0.02	0.01	0.01	0.01	0.03					0.01	0.01	0.03	0.03	0.02	0.03								0.43	0.033
5	0.06	0.02	0.04	0.03	0.04	0.04	0.03	0.05	0.06		0.07	0.06	0.07	0.03											0.60	0.046
6				0.04	0.06	0.03	0.10	0.04	0.03	0.06	0.07	0.10	0.05	0.04	0.08	0.07	0.03	0.04	0.05	0.04	0.03	0.05	0.06	0.04	1.11	0.053
7	0.05	0.04	0.06	0.06	0.04	0.03	0.08	0.04	0.08	0.07	0.10	0.03	0.12	0.08	0.12	0.17	0.32	0.07	0.06	0.06	0.05	0.03	0.02	0.27	2.01	0.085
8	0.10	0.04	0.09	0.04	0.03	0.04	0.06	0.22	0.10	0.13	0.12	0.04													1.01	0.084
9											0.30	0.12			0.02	0.04	0.03								0.51	0.102
10																										
11											0.40	0.05	0.04	0.06											0.13	0.043
12																0.10	0.10		0.04	0.03	0.04	0.06	0.05	0.03	0.45	0.056
13	0.08	0.03	0.03	0.02	0.05	0.08	0.03				0.10	0.07	0.13	0.08	0.01	0.04	0.06	0.20	0.05	0.06	0.05	0.04	0.04	0.03	1.29	0.061
14	0.04	0.03													0.08	0.16	0.08	0.02	0.03	0.04	0.03	0.03	0.02	0.03	0.59	0.049
15	0.14	0.04											0.06	0.06	0.02	0.01	0.03	0.12	0.02	0.03					0.55	0.055
16																0.03	0.07	0.03	0.06	0.03	0.03	0.02	0.03	0.04	0.34	0.038
17	0.03	0.03	0.02								0.07	0.08	0.05	0.02	0.03	0.07					0.03	0.03	0.02	0.04	0.56	0.043
18	0.03	0.01	0.01	0.01	0.03	0.02	0.03	0.05	0.06	0.03	0.04	0.04	0.03	0.03	0.01	0.04	0.01	0.02	0.03	0.01	0.02	0.05	0.02	0.01	0.72	0.030
19	0.03	0.04	0.04	0.02	0.08	0.10	0.04	0.06	0.04	0.03	0.04	0.03	0.09												0.64	0.049
20											0.04	0.04	0.04	0.06	0.07	0.03	0.03	0.06	0.03	0.02	0.04	0.08	0.08	0.04	0.63	0.045
21	0.04	0.03	0.03	0.02	0.16	0.05	0.04	0.03	0.03	0.02	0.03	0.08	0.02	0.06	0.01	0.04	0.09	0.02	0.06	0.05	0.04	0.03	0.05	0.07	1.10	0.046
22	0.10	0.08	0.03	0.04	0.03	0.02	0.03	0.06	0.08	0.09	0.12	0.06	0.05	0.06	0.08	0.06	0.11	0.06	0.05	0.04	0.02	0.03	0.01	0.02	1.33	0.055
23	0.02	0.04	0.09	0.03	0.02	0.02	0.01	0.02	0.08	0.05	0.03	0.04	0.03	0.01											0.49	0.035
24												0.07	0.05	0.09	0.06	0.05	0.12	0.06	0.09	0.03	0.01	0.03	0.02	0.02	0.70	0.054
25	0.03	0.15	0.03	0.16	0.03	0.04	0.03	0.04	0.06	0.04	0.08	0.09	0.04	0.05	0.06	0.08	0.07	0.04	0.06	0.04	0.02	0.05	0.03	0.02	1.34	0.056
26	0.04	0.03	0.02	0.05	0.03	0.02	0.02		0.02	0.03	0.07	0.08	0.04	0.07	0.02	0.04	0.02	0.10	0.14	0.04	0.12	0.04	0.08	0.02	1.14	0.050
27	0.03	0.03	0.02	0.02	0.04	0.02	0.02	0.04	0.02	0.05	0.07	0.03	0.06	0.04	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.04	0.01	0.03	0.65	0.027
28	0.05	0.02	0.04	0.04	0.07	0.03	0.04	0.03	0.05	0.05	0.07	0.02	0.02	0.02	0.04	0.03	0.02	0.07	0.05	0.02	0.12	0.02	0.04	0.02	0.86	0.033
29	0.04	0.02	0.03	0.02	0.03	0.04	0.02	0.02	0.03	0.04	0.06	0.07	0.02	0.07	0.03	0.02	0.04	0.03	0.02	0.04	0.03	0.04	0.02	0.06	0.78	0.033
30	0.04	0.03	0.02	0.02	0.02	0.01	0.02	0.04	0.06	0.04	0.03	0.02	0.04	0.02											0.41	0.029
31																										
total	1.08	0.85	0.70	0.73	0.85	0.70	0.74	1.01	1.03	0.93	1.93	1.67	1.49	1.12	10.2	1.50	2.48	1.94	1.36	0.98	1.21	0.86	0.70	1.18	28.06	0.057
Mean	0.054	0.040	0.037	0.038	0.045	0.037	0.039	0.063	0.061	0.058	0.084	0.067	0.062	0.053	0.049	0.065	0.113	0.102	0.068	0.049	0.061	0.043	0.035	0.059	0.058	

(E. W.) 1942. April

day \ hour	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2																		0.07	0.05	0.06	0.06		0.03	0.03	0.30	0.050
3	0.02	0.01	0.05	0.02	0.04	0.07	0.03	0.02	0.02	0.02	0.02	0.01	0.02	0.03	0.02	0.08	0.06	0.03	0.01	0.02	0.03	0.03	0.02	0.06	0.77	0.032
4	0.09	0.04	0.02	0.01	0.03	0.02	0.03	0.03	0.25	0.04	0.02						0.06	0.03	0.04	0.04	0.01	0.04	0.03	0.03	0.89	0.047
5	0.02	0.02	0.03	0.02	0.03	0.06	0.02	0.03	0.07	0.03	0.06	0.01	0.02	0.03	0.06	0.05	0.06	0.07	0.05	0.12	0.03	0.06	0.04	0.14	1.13	0.047
6	0.03	0.01	0.02	0.01	0.06	0.03	0.16	0.05	0.06	0.07	0.05	0.06	0.02	0.02	0.01	0.02	0.01	0.02	0.03	0.02	0.02	0.02	0.03	0.02	0.91	0.038
7	0.01	0.01	0.03	0.02	0.02	0.01	0.02	0.03	0.01	0.03	0.01	0.02	0.02	0.02			0.02	0.03	0.01	0.03	0.02	0.02	0.02	0.02	0.49	0.022
8	0.02	0.06									0.01	0.07	0.02	0.07	0.03	0.06	0.02	0.04	0.01	0.02	0.02	0.01	0.03	0.02	0.54	0.034
9	0.12	0.08	0.20	0.06	0.01	0.08	0.06	0.07	0.08	0.09	0.16	0.16	0.04	0.04	0.03	0.10	0.08	0.02	0.01	0.05	0.06	0.03	0.06	0.02	1.74	0.073
10	0.10	0.06	0.02	0.01	0.02	0.04	0.03	0.10	0.08	0.07	0.01	0.05	0.06	0.08	0.05	0.01	0.05	0.03	0.03	0.01	0.02	0.03	0.01	0.02	1.05	0.044
11	0.03	0.02	0.02	0.01	0.02	0.02	0.03	0.04	0.08	0.06	0.06	0.07	0.10	0.04	0.04	0.09	0.14	0.13	0.02	0.03	0.02	0.05	0.05	0.03	1.20	0.050
12	0.02	0.02	0.03	0.01	0.01	0.02	0.02	0.02	0.03	0.04	0.04	0.03	0.02	0.03	0.03	0.02	0.01	0.02	0.02	0.02	0.03	0.01	0.03	0.02	0.55	0.023
13	0.02	0.03		0.02	0.01	0.02	0.03	0.01	0.04	0.07	0.03	0.05	0.03	0.04	0.04	0.07	0.04	0.05	0.03	0.05	0.02	0.03	0.02	0.05	0.86	0.037
14	0.01	0.03	0.02	0.01	0.02	0.02	0.04	0.06	0.10	0.08	0.23	0.12	0.03	0.03	0.02	0.04	0.01	0.04	0.02	0.01	0.06	0.03	0.02	0.03	1.08	0.045
15	0.02	0.01	0.03	0.03	0.01	0.03	0.01	0.02	0.05	0.03	0.06	0.02	0.03	0.02	0.01	0.03	0.02	0.03	0.02	0.06	0.03	0.03	0.02	0.02	0.70	0.029
16	0.02	0.01	0.02	0.01	0.02	0.01	0.06	0.07	0.04		0.03	0.02	0.03	0.03	0.03	0.01	0.03	0.01	0.02	0.01	0.03	0.02	0.04	0.02	0.61	0.027
17	0.02	0.02	0.07	0.01	0.03	0.05	0.01	0.03	0.05	0.03	0.06	0.12	0.08	0.05	0.04	0.06	0.06	0.07	0.01	0.03	0.01	0.01	0.02	0.01	1.04	0.043
18	0.01	0.03	0.01	0.01	0.02	0.04	0.02	0.04	0.06	0.07	0.12	0.10	0.03	0.06	0.07	0.01	0.20	0.19	0.06	0.03	0.01	0.02	0.02	0.03	1.35	0.056
19	0.02	0.03	0.01	0.03	0.05	0.04	0.02		0.07	0.08	0.12	0.11	0.05	0.03	0.06	0.10	0.05	0.10	0.03	0.03	0.04	0.03	0.02	0.04	1.13	0.049
20	0.03	0.02	0.03	0.03	0.04	0.03	0.02	0.06	0.03	0.05	0.05	0.06	0.05	0.02	0.06	0.04	0.02	0.03	0.02	0.05	0.02	0.02	0.01	0.01	0.80	0.033
21	0.03	0.14	0.01	0.02	0.02	0.01	0.02	0.03	0.04	0.06	0.04	0.02	0.06	0.05	0.04	0.02	0.05	0.03	0.07	0.01	0.02	0.05	0.03	0.02	0.92	0.038
22	0.03	0.02	0.03	0.02	0.03	0.07		0.07	0.06	0.08	0.04	0.04	0.04	0.03	0.03	0.06	0.05	0.06	0.04	0.04	0.03	0.02	0.03	0.02	0.91	0.041
23	0.02	0.03	0.02	0.06	0.02	0.02	0.03	0.04	0.06	0.07	0.10	0.10	0.07	0.06	0.14	0.06	0.01	0.12	0.05	0.08	0.04	0.05	0.02	0.04	1.31	0.055
24	0.02	0.02	0.07	0.03	0.05	0.05	0.06	0.10	0.20	0.16	0.09	0.26	0.06	0.04	0.06	0.09	0.09	0.07	0.01	0.06	0.04	0.04	0.03	0.01	1.79	0.075
25	0.04	0.03	0.03	0.04	0.01	0.06	0.01																		0.28	0.040
26																										
27															0.02	0.02	0.06	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.25	0.025
28	0.02	0.03	0.02	0.02	0.03	0.05	0.02	0.03	0.02	0.06	0.06	0.08	0.02	0.06	0.02	0.05	0.06	0.06	0.02	0.01	0.03	0.02	0.02	0.01	0.83	0.035
29	0.03	0.02	0.01	0.01	0.02	0.03	0.03	0.01	0.03	0.06	0.04	0.01	0.04	0.05	0.04	0.05	0.02	0.02	0.01	0.02	0.02	0.01	0.01	0.01	0.68	0.028
30	0.02	0.03	0.01	0.02	0.04	0.02	0.04	0.05	0.03	0.03	0.02	0.06	0.02	0.03											0.42	0.030
31																										
Σ	0.88	0.89	0.84	0.60	0.79	0.99	0.90	1.07	1.53	1.38	1.62	1.68	0.96	0.96	0.98	1.20	1.28	1.40	0.74	0.93	0.80	0.72	0.67	0.77	24.56	0.041
Mean	0.034	0.034	0.035	0.024	0.032	0.039	0.038	0.047	0.064	0.060	0.065	0.070	0.040	0.040	0.043	0.052	0.051	0.054	0.028	0.036	0.031	0.029	0.026	0.030	0.042	

Hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1											0.02	0.02	0.08	0.02	0.01	0.01	0.06	0.01	0.03	0.02	0.04	0.04	0.03	0.03	0.42	0.030
2	0.02	0.01	0.01	0.02	0.01	0.03	0.02	0.06	0.04	0.07	0.12					0.06	0.07	0.06		0.02	0.06	0.03			0.71	0.042
3																										
4												0.05	0.03	0.02	0.05	0.06	0.03	0.02	0.04	0.03	0.02	0.03	0.02	0.01	0.41	0.032
5	0.02	0.01	0.08	0.08	0.03	0.02	0.06	0.02	0.02	0.02	0.03	0.06	0.05	0.02	0.10	0.05	0.03		0.04				0.03		0.77	0.044
6			0.02				0.07	0.06	0.03	0.05	0.03	0.02	0.04												0.32	0.040
7																0.03	0.06	0.02	0.03	0.04	0.02	0.01	0.03	0.02	0.26	0.029
8	0.03	0.03	0.04	0.03	0.03	0.03	0.02	0.02	0.03	0.04	0.03	0.04	0.03	0.02	0.04	0.02	0.02	0.01	0.02	0.02	0.02	0.02	0.03	0.01	0.63	0.026
9	0.01	0.02	0.02	0.01	0.02	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.03	0.03	0.02	0.03	0.02	0.04	0.03	0.03	0.03	0.03	0.04	0.03	0.58	0.024
10	0.01	0.03	0.02	0.02	0.03	0.02	0.03	0.01	0.02	0.03	0.02	0.02	0.02	0.06	0.04	0.02	0.03	0.06	0.04	0.03	0.03	0.01	0.02	0.04	0.66	0.028
11	0.03	0.04	0.04	0.10	0.06											0.03	0.03	0.03	0.05	0.02	0.01	0.01	0.04	0.06	0.55	0.039
12	0.05	0.02	0.01	0.03	0.06	0.05	0.03	0.04	0.03	0.02	0.02	0.01	0.02	0.01	0.03	0.06	0.03	0.08							0.60	0.033
13											0.06	0.05	0.03	0.04	0.01	0.03	0.03	0.03	0.02	0.03	0.03	0.02	0.06	0.02	0.46	0.033
14	0.03	0.02	0.01	0.03	0.02	0.03	0.03	0.03	0.01	0.03	0.03	0.06	0.05	0.04	0.05	0.10	0.03	0.03	0.01	0.02	0.02	0.01	0.03	0.02	0.74	0.031
15	0.02	0.02	0.04	0.03	0.02	0.02	0.04	0.03	0.05	0.03	0.03	0.03	0.02	0.06	0.01	0.01	0.01	0.01	0.02	0.04	0.05	0.02	0.03	0.05	0.69	0.029
16	0.03	0.10	0.03	0.02	0.04	0.20	0.06	0.05	0.06	0.04	0.06	0.04	0.04	0.03	0.04	0.06	0.08	0.12	0.08	0.04	0.08	0.04	0.03	0.03	1.40	0.058
17	0.02	0.05	0.04	0.02	0.06	0.03	0.03	0.02	0.10	0.10	0.10	0.08	0.09	0.07	0.05	0.05	0.06	0.06	0.05	0.05					1.13	0.057
18																										
19										0.06	0.04	0.06	0.03	0.05	0.05	0.04	0.04	0.03	0.08	0.03	0.04	0.03	0.03	0.04	0.65	0.043
20	0.04	0.04	0.03	0.04	0.13	0.06	0.05	0.03	0.12	0.07	0.06	0.05	0.05	0.05	0.03	0.04	0.03	0.03	0.02	0.03	0.04	0.03	0.04	0.03	1.14	0.048
21	0.08	0.06	0.02	0.02	0.03	0.04	0.06	0.07	0.04	0.08	0.12	0.02	0.05	0.03	0.06	0.08	0.03	0.06	0.02	0.02	0.04	0.06	0.02	0.02	1.13	0.047
22	0.18	0.02	0.02	0.02	0.02	0.02	0.02	0.01	0.02	0.03	0.03	0.02	0.01	0.03	0.02	0.05	0.12	0.06	0.03	0.02	0.02	0.03	0.02	0.01	0.83	0.035
23	0.03	0.04									0.06	0.03	0.04	0.03	0.03	0.04	0.06	0.02	0.04	0.02	0.02	0.02	0.01	0.01	0.50	0.031
24	0.02	0.03	0.02	0.01	0.02	0.04	0.03	0.06	0.02	0.06	0.12	0.15			0.02	0.03	0.03	0.04	0.02	0.02	0.03	0.02	0.03	0.08	0.90	0.041
25	0.03	0.04	0.04	0.03	0.02	0.03	0.04	0.06	0.06	0.07	0.08	0.03	0.04	0.03	0.04	0.02	0.06	0.03	0.04	0.03	0.04	0.02	0.02	0.03	0.93	0.039
26	0.03	0.04	0.03	0.03	0.04	0.03	0.05	0.07	0.03	0.06	0.02	0.03	0.03	0.03	0.02	0.03	0.02	0.02	0.03	0.02	0.01	0.02	0.01	0.03	0.73	0.030
27	0.04	0.03	0.02	0.02	0.02	0.01	0.02	0.02	0.04	0.03	0.01	0.03	0.04	0.04	0.03	0.02	0.04	0.03	0.02	0.05	0.03	0.02	0.03	0.04	0.68	0.028
28	0.03	0.02	0.04	0.03	0.14	0.03	0.04	0.14	0.16	0.08	0.10	0.08	0.10	0.03	0.08	0.06	0.05	0.10	0.04	0.03	0.03	0.03	0.03	0.01	1.48	0.062
29	0.05	0.03	0.02	0.01	0.03	0.04	0.03	0.06	0.04	0.06	0.06	0.08	0.06	0.06	0.10	0.04	0.06	0.08	0.08	0.07	0.04	0.03	0.02	0.01	1.16	0.048
30	0.02	0.03	0.03	0.03	0.03	0.02	0.04	0.02	0.04	0.02	0.01	0.01	0.03	0.06	0.03	0.02	0.03	0.10	0.04	0.02	0.01	0.02	0.01	0.03	0.70	0.029
31	0.02	0.02	0.03	0.04	0.03	0.07	0.03	0.02	0.03	0.01												0.03	0.02	0.02	0.37	0.028
total	0.84	0.75	0.66	0.67	0.89	0.84	0.81	0.92	1.00	1.08	1.30	1.09	1.01	0.86	0.96	1.09	1.16	1.18	0.92	0.75	0.76	0.63	0.68	0.68	21.53	0.037
Mean	0.037	0.033	0.029	0.030	0.040	0.040	0.037	0.042	0.045	0.047	0.052	0.044	0.042	0.037	0.040	0.040	0.043	0.045	0.037	0.030	0.032	0.025	0.027	0.028	0.038	

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.06	0.03	0.02	0.03	0.05	0.03	0.02	0.01	0.03	0.03	0.04	0.03	0.02	0.03	0.02	0.03	0.13	0.02	0.03	0.04	0.02	0.02	0.03	0.02	0.79	0.033
2	0.01	0.01	0.02	0.02	0.06	0.06	0.03	0.02	0.04	0.02	0.02	0.03	0.03	0.05	0.02	0.01	0.03	0.04	0.05	0.04	0.02	0.03	0.03	0.04	0.73	0.030
3	0.04	0.02	0.02	0.03	0.02	0.03	0.03	0.02	0.02	0.01	0.05	0.02	0.01	0.02	0.02	0.03	0.03	0.03	0.05	0.05	0.02	0.01	0.04	0.03	0.65	0.027
4	0.06	0.02	0.12	0.03	0.03	0.02	0.05	0.06	0.04	0.03	0.04	0.07	0.03	0.02	0.04	0.05	0.10	0.03	0.04	0.02	0.01	0.02	0.02	0.05	1.00	0.042
5	0.04	0.03		0.01	0.18	0.03	0.04	0.03	0.04	0.08	0.03	0.08	0.12	0.14	0.12	0.07	0.03	0.04	0.03	0.05	0.04	0.03	0.04	0.04	1.34	0.058
6	0.03	0.02	0.02	0.01	0.03	0.03	0.02	0.02	0.04	0.05	0.09				0.04	0.03	0.02	0.03	0.03	0.04	0.04	0.25	0.03	0.04	0.91	0.043
7	0.04	0.10	0.03	0.04	0.02	0.03	0.04	0.03	0.03																0.36	0.040
8										0.06	0.04	0.03	0.02	0.06	0.04	0.04	0.02	0.03	0.02	0.02	0.03	0.02	0.01	0.01	0.45	0.030
9	0.02	0.03	0.02	0.01	0.03	0.03	0.03	0.02	0.02	0.03	0.02	0.04	0.03	0.05	0.04	0.04	0.03	0.02	0.04	0.08	0.02	0.02	0.01	0.01	0.69	0.029
10	0.01	0.02	0.02	0.01	0.02	0.02	0.02	0.01	0.01	0.02	0.02	0.03	0.02	0.03	0.02	0.02	0.03	0.02	0.01	0.02	0.04	0.02	0.02	0.01	0.47	0.020
11	0.01	0.02	0.02	0.03	0.04	0.02	0.04	0.02	0.02																0.22	0.024
12										0.07	0.07	0.05	0.03	0.05	0.08	0.09	0.16	0.04	0.22	0.03	0.03	0.05	0.03	1.00	0.071
13	0.04	0.07	0.03	0.06	0.03	0.04	0.02	0.05	0.02	0.03	0.09	0.21	0.06	0.14	0.07	0.06	0.03	0.02	0.03	0.03	0.05	0.04	0.05	0.02	1.29	0.054
14	0.04	0.44	0.04	0.04	0.03	0.06	0.06	0.10	0.08	0.06	0.22	0.16	0.06	0.10	0.05	0.10	0.08	0.12	0.09	0.06	0.08	0.07	0.07	0.10	2.31	0.096
15	0.04	0.06	0.04	0.05	0.04	0.06	0.02	0.06	0.12	0.06	0.06	0.06	0.14	0.14	0.04	0.10	0.03	0.05	0.05	0.03	0.06	0.04	0.03	0.08	1.45	0.060
16	0.03	0.03	0.02	0.03	0.03	0.09	0.03	0.08	0.05	0.06	0.02	0.03	0.02	0.06	0.04	0.02	0.04	0.03	0.01	0.10	0.03	0.02	0.03	0.03	0.93	0.039
17	0.04	0.03	0.02	0.02	0.04	0.06	0.02	0.02	0.02	0.07	0.06	0.04	0.05	0.06	0.06	0.04	0.14	0.04	0.03	0.02	0.03	0.04	0.04	0.03	1.08	0.045
18										0.08	0.11	0.12	0.13	0.05	0.06	0.05	0.13	0.10	0.05	0.08	0.04	0.05	0.04	0.03	1.12	0.075
19	0.03	0.03	0.02	0.04	0.02	0.05	0.02	0.02	0.04		0.06	0.04	0.05	0.03	0.02	0.04	0.05	0.03	0.02	0.03	0.02	0.06	0.02	0.04	0.78	0.034
20	0.02	0.03	0.04	0.02	0.02	0.04	0.04	0.06	0.05	0.07	0.10	0.16	0.09	0.08	0.06	0.08	0.07	0.08	0.06	0.08	0.03	0.02	0.03	0.03	1.36	0.057
21	0.03	0.02	0.02	0.04	0.02	0.02	0.04	0.03	0.04	0.02	0.04	0.04	0.02	0.06	0.02	0.06	0.04	0.06	0.04	0.03	0.04	0.02	0.02	0.04	0.81	0.034
22	0.10	0.08	0.06	0.05	0.14	0.03	0.02	0.02	0.03	0.03	0.02	0.04	0.03	0.04	0.03	0.02	0.05	0.04	0.02	0.03	0.04	0.02	0.03	0.02	0.99	0.041
23	0.03	0.04	0.02	0.03							0.05	0.05	0.06	0.05	0.04	0.06	0.06	0.11	0.14	0.06	0.04	0.02	0.03	0.08	0.97	0.054
24	0.02	0.02	0.16	0.04	0.03	0.04	0.05	0.05	0.04	0.11															0.56	0.056
25																										
26																										
27																										
28																										
29																										
30																										
31																										
total	0.74	1.15	0.76	0.71	0.88	0.78	0.66	0.70	0.78	0.92	1.25	1.35	1.04	1.24	0.90	1.03	1.23	1.10	0.88	1.13	0.73	0.85	0.67	0.78	22.26	0.046
Mean	0.024	0.037	0.025	0.023	0.028	0.025	0.021	0.023	0.025	0.030	0.040	0.045	0.034	0.040	0.029	0.033	0.040	0.035	0.029	0.036	0.024	0.027	0.022	0.025	0.046	

(E. W.) 1942, July

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1										0.06	0.08	0.06	0.10	0.12	0.08	0.18	0.06	0.07	0.17	0.09	0.05	0.10	0.03	0.02	1.27	0.085
2	0.05	0.04	0.04	0.06	0.06	0.05	0.06	0.07	0.07	0.08	0.04	0.04	0.05	0.08	0.05	0.02	0.02	0.03	0.03	0.01	0.05	0.01	0.01	0.02	1.02	0.043
3	0.02	0.08	0.07	0.09	0.10	0.07	0.10	0.15	0.10	0.07	0.08	0.11	0.03	0.03	0.02	0.07	0.10	0.06	0.06	0.02	0.06	0.07	0.02	0.03	1.61	0.067
4	0.03	0.03	0.02	0.03	0.05	0.02	0.02	0.04	0.12	0.06	0.07	0.07	0.06	0.06	0.07	0.02	0.06	0.09	0.17						1.07	0.056
5											0.10	0.08	0.05	0.02	0.01	0.08	0.09	0.06	0.02	0.02	0.01	0.02	0.02	0.04	0.62	0.044
6	0.06	0.02	0.02	0.02	0.10	0.03	0.03						0.08	0.08	0.10	0.07	0.07	0.06	0.05	0.04	0.03	0.02	0.02	0.02	0.92	0.048
7	0.04	0.04						0.08		0.10															0.26	0.065
8												0.03	0.07	0.16	0.25	0.20	0.12	0.10	0.17	0.09	0.08	0.02	0.02	0.02	1.34	0.103
9	0.01	0.02	0.02	0.03	0.10	0.01	0.06	0.17	0.22	0.18	0.20	0.06	0.11	0.03	0.08	0.10	0.12	0.05	0.05	0.03	0.02	0.01	0.01	0.01	1.71	0.072
10	0.07	0.06	0.07	0.06	0.02	0.05	0.03	0.07	0.18	0.12	0.11	0.10	0.10	0.09	0.18	0.09	0.02	0.02	0.03	0.09	0.01	0.01	0.05	0.03	1.64	0.068
11	0.03	0.06	0.01	0.01	0.01	0.02	0.02	0.03	0.08	0.07	0.26	0.23	0.05	0.11	0.15	0.10	0.08	0.03	0.01	0.01	0.02	0.02	0.06	0.02	1.49	0.062
12	0.06	0.04	0.12	0.16	0.15	0.06	0.08	0.06	0.06	0.10	0.08	0.03													1.00	0.083
13										0.07	0.06	0.08	0.08	0.09	0.02	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.54	0.036
14											0.03	0.03	0.02												0.08	0.027
15													0.03	0.04	0.15	0.04	0.06	0.10	0.12	0.09	0.03				0.66	0.073
16												0.10	0.18	0.10	0.12	0.11	0.02	0.06							0.69	0.099
17												0.10	0.05	0.06	0.05	0.03	0.02	0.20	0.02	0.12	0.20	0.06	0.02	0.02	0.95	0.073
18	0.02	0.01	0.03	0.02	0.01	0.03	0.02	0.02	0.03	0.02	0.01	0.03	0.05	0.05	0.02	0.03	0.02	0.03	0.03	0.01	0.06	0.02	0.03	0.02	0.67	0.028
19	0.02	0.02	0.03	0.02	0.02	0.01	0.02	0.01	0.02	0.01	0.03	0.02	0.03	0.02	0.02	0.01	0.04	0.06	0.05	0.06	0.02	0.01	0.03	0.03	0.61	0.025
20	0.03	0.02	0.02	0.02	0.02	0.02	0.01	0.02	0.01	0.03	0.02	0.02	0.05	0.03	0.02	0.02	0.06	0.07	0.05	0.06	0.02	0.01	0.03	0.02	0.68	0.028
21	0.03	0.01	0.03	0.04	0.10	0.07	0.07	0.06	0.10	0.08	0.12	0.15	0.08	0.02	0.04	0.09	0.14	0.06	0.08	0.05	0.03	0.02	0.02	0.02	1.56	0.065
22	0.02	0.03	0.02	0.03	0.05	0.01	0.05	0.03	0.05	0.03	0.07	0.05	0.09	0.02	0.05	0.01	0.01	0.02	0.02	0.05	0.02	0.03	0.04	0.02	0.85	0.035
23	0.02	0.01	0.02	0.05	0.02	0.05	0.02	0.04	0.06	0.07	0.05	0.04	0.12	0.04	0.02	0.04	0.02	0.03	0.02	0.02	0.02	0.01	0.02	0.02	0.78	0.033
24	0.03	0.10	0.02	0.10	0.03	0.04	0.03	0.03	0.07	0.03	0.08	0.09	0.05	0.02	0.08	0.07	0.04	0.10	0.01	0.02	0.03	0.03	0.03	0.02	1.18	0.049
25	0.18	0.10	0.03	0.03	0.03	0.03	0.04	0.04	0.06	0.07															0.61	0.061
26																0.12	0.02	0.02	0.05	0.05	0.03	0.02	0.02	0.02	0.35	0.039
27	0.02	0.07	0.03	0.02	0.03	0.02	0.10	0.08	0.09	0.12	0.06	0.18	0.12	0.15	0.16	0.05	0.18	0.17	0.15	0.03	0.03	0.02	0.02	0.05	1.98	0.083
28	0.06	0.02	0.02	0.03	0.02	0.07	0.03	0.03	0.05	0.05	0.11	0.09	0.12	0.14	0.06	0.15	0.03	0.04	0.03	0.03	0.04	0.07	0.06	0.02	1.53	0.063
29	0.02	0.02	0.01	0.01	0.03	0.04	0.05	0.03	0.05	0.11	0.15	0.24	0.08	0.01	0.05	0.17	0.07	0.01	0.13	0.04	0.03	0.02	0.01	0.02	1.46	0.061
30	0.03	0.06	0.02	0.02	0.04	0.01	0.03	0.04	0.03	0.10	0.11	0.15	0.10	0.02	0.03	0.07	0.09	0.07	0.06	0.05	0.04	0.02	0.03	0.02	1.27	0.053
31	0.02	0.02	0.06	0.19	0.03	0.05	0.06	0.09	0.10	0.11	0.12	0.09	0.05	0.09	0.11	0.05	0.05	0.10	0.12	0.10	0.05	0.08	0.04	0.13	1.91	0.080
total	0.87	0.88	0.91	1.04	1.00	0.81	0.93	1.35	1.57	1.87	2.03	2.37	2.00	1.71	1.91	1.00	1.59	1.73	1.79	1.20	1.05	0.73	0.66	0.66	32.31	0.058
Mean	0.010	0.010	0.031	0.050	0.038	0.039	0.044	0.060	0.070	0.084	0.068	0.082	0.072	0.066	0.073	0.074	0.059	0.064	0.069	0.048	0.042	0.030	0.028	0.028	0.056	

Hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.02	0.03	0.03	0.04	0.05	0.02	0.07	0.04			0.08	0.06	0.03	0.04	0.03	0.08	0.04	0.04	0.02	0.02	0.01	0.02	0.02	0.03	0.82	0.037
2	0.04	0.02	0.03	0.02	0.03	0.06	0.02	0.03	0.08	0.06	0.12	0.10	0.10	0.20	0.08	0.07	0.12	0.06	0.02	0.08	0.03	0.02	0.09	0.03	1.51	0.063
3	0.02	0.06	0.02	0.03	0.04	0.08	0.04	0.08	0.04	0.17	0.19	0.12	0.13	0.04	0.06	0.03	0.08	0.03	0.13	0.04	0.02	0.03	0.04	0.04	1.56	0.065
4	0.07	0.03	0.02	0.04	0.04	0.05	0.06	0.07	0.03	0.08	0.10	0.09	0.02	0.04	0.03	0.02	0.03	0.01	0.06	0.02	0.07	0.07	0.03	0.01	1.39	0.058
5	0.02	0.02	0.02	0.03	0.10	0.15				0.01	0.03	0.04	0.05	0.07	0.06	0.06	0.05	0.06	0.08	0.04	0.03	0.14	0.01	0.02	1.10	0.052
6	0.02	0.04	0.02	0.01	0.01	0.02	0.06	0.07	0.08	0.06	0.02	0.02	0.03	0.02	0.03	0.03	0.04	0.02	0.01	0.06		0.03	0.01	0.10	0.81	0.035
7	0.02	0.02	0.01	0.04	0.02	0.02	0.04	0.01	0.04	0.03	0.05	0.02	0.01	0.03	0.01	0.04	0.06	0.11	0.06	0.07	0.02	0.07	0.04	0.03	1.03	0.043
8	0.02	0.02	0.16	0.04	0.03	0.02	0.05	0.04	0.08	0.03	0.02	0.02	0.02	0.01	0.05	0.05	0.10	0.02	0.04	0.01	0.01	0.02	0.01	0.02	0.89	0.037
9	0.01	0.01	0.02	0.01	0.02	0.02	0.02	0.01	0.03	0.09		0.02	0.04		0.03	0.03	0.08	0.04	0.04	0.04	0.04	0.01			0.61	0.031
10																										
11																										
12																										
13																										
14												0.12	0.06	0.04	0.05	0.04	0.02	0.02	0.04	0.02	0.02	0.01	0.02	0.02	0.48	0.037
15	0.04	0.02	0.02	0.03	0.01	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.01	0.03	0.02	0.01	0.04	0.04	0.06	0.02	0.01	0.01	0.04	0.02	0.55	0.023
16	0.02	0.01	0.02	0.02	0.01	0.02	0.04	0.01	0.08	0.03	0.04	0.17	0.04	0.02	0.03	0.07	0.02	0.02	0.04	0.07	0.02	0.02	0.03	0.01	0.80	0.033
17	0.02	0.02	0.05	0.05	0.02	0.07	0.12	0.30	0.36	0.25	0.03	0.06	0.10	0.15	0.03	0.02	0.03	0.10	0.09	0.08	0.05	0.04	0.02	0.03	2.00	0.083
18	0.04	0.01	0.03	0.04	0.04	0.04	0.03	0.02	0.03	0.05	0.02	0.05								0.03	0.03	0.02	0.03	0.03	0.35	0.032
19	0.03	0.02																						0.01	0.06	0.020
20	0.02	0.05	0.02	0.02	0.02	0.02	0.01	0.02	0.12	0.04			0.02	0.01	0.02	0.03	0.01	0.01	0.05	0.04	0.01	0.08	0.04	0.02	0.71	0.032
21	0.02	0.02	0.02	0.06																					0.12	0.030
22																										
23										0.06	0.04	0.15	0.11	0.10	0.19	0.04	0.12	0.08	0.02	0.04	0.02	0.02	0.01	0.06	1.06	0.071
24	0.06	0.01	0.01	0.02	0.03	0.04	0.05	0.01	0.17	0.05	0.12	0.07	0.12	0.15	0.02	0.01	0.06	0.10	0.03	0.04	0.02	0.02	0.01	0.03	1.40	0.058
25	0.03	0.02	0.02	0.03	0.02	0.06	0.10	0.05	0.23	0.08															0.46	0.064
26																										
27																										
28																										
29										0.08	0.04	0.04	0.03	0.02	0.02	0.01	0.06	0.02	0.02	0.02	0.02	0.01	0.03	0.03	0.56	0.037
30		0.06	0.02	0.01	0.01	0.03	0.03	0.02	0.07	0.04	0.02	0.03	0.03	0.02	0.01	0.03	0.01	0.02	0.07	0.01	0.03				0.73	0.024
31											0.02	0.03	0.06	0.03	0.02	0.03	0.06	0.02	0.03	0.01	0.03	0.03	0.02	0.01	0.45	0.015
Total	0.83	0.68	0.53	0.83	0.63	1.13	0.85	0.85	1.35	1.27	0.77	1.27	1.01	0.98	0.70	0.75	1.05	0.75	0.80	0.76	0.45	0.67	0.50	0.61	14.38	0.042
Mean	0.027	0.023	0.017	0.027	0.020	0.036	0.027	0.027	0.044	0.041	0.025	0.041	0.033	0.032	0.023	0.024	0.034	0.024	0.026	0.025	0.015	0.022	0.016	0.020	0.464	0.036

(E. W.) 1942, Sept

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.03	0.02	0.01	0.02	0.03	0.03	0.04	0.04	0.05	0.06				0.02	0.01	0.02	0.01	0.05	0.02	0.02	0.06	0.01	0.03	0.02	0.66	0.031
2	0.01			0.01	0.02	0.03	0.04	0.04	0.03	0.01	0.02	0.05	0.09		0.06	0.05	0.06	0.02	0.02	0.03	0.02	0.01	0.01	0.04	0.73	0.035
3	0.03	0.02	0.03	0.01	0.03	0.01	0.06	0.06	0.03	0.05	0.04	0.01							0.03	0.03	0.01	0.02	0.02	0.02	0.63	0.035
4	0.03	0.02	0.01	0.05	0.02	0.02	0.03	0.04	0.05	0.04															0.31	0.031
5																										
6																										
7																										
8													0.04	0.03	0.02	0.01	0.02	0.03	0.03	0.02	0.02	0.02	0.04	0.02	0.30	0.025
9	0.02	0.02	0.01	0.02	0.03	0.03	0.02	0.01	0.03	0.10	0.07	0.01	0.07	0.03	0.07	0.02	0.02	0.02	0.04	0.01	0.06	0.02	0.02	0.01	0.70	0.029
10	0.02	0.02	0.02	0.12	0.02	0.04	0.05	0.07	0.02	0.03	0.06	0.05	0.02	0.04	0.03	0.01	0.01	0.01	0.02	0.04	0.05	0.03	0.02	0.01	0.81	0.034
11	0.02	0.01	0.01	0.02	0.01	0.01	0.02	0.03	0.04	0.02	0.04	0.05													0.28	0.023
12													0.23	0.23	0.16	0.23	0.14	0.08	0.07	0.07	0.03	0.03	0.10	0.06	1.43	0.119
13	0.02	0.06	0.02	0.02	0.03	0.03	0.04	0.04	0.10	0.12	0.08	0.01	0.07	0.12	0.07	0.10	0.04	0.12	0.11	0.01	0.02	0.04	0.02	0.01	1.36	0.057
14	0.05	0.01	0.06	0.01	0.08	0.03	0.02	0.13	0.11	0.18	0.13	0.10	0.15	0.12	0.12	0.13	0.14	0.21	0.08	0.06	0.04	0.03	0.03	0.02	2.04	0.085
15	0.27	0.04	0.03	0.03	0.02	0.02	0.03	0.04	0.10	0.12	0.10	0.20	0.06	0.15	0.10	0.12	0.06	0.14	0.02	0.06	0.12	0.06	0.04	0.10	2.03	0.084
16	0.01	0.03	0.07	0.01	0.01	0.03	0.02	0.10	0.02	0.03	0.01	0.06	0.02	0.10	0.05	0.03	0.44	0.06	0.03	0.12	0.02	0.06	0.02	0.01	1.10	0.046
17	0.05	0.03	0.02	0.02	0.01	0.01	0.06	0.16	0.19	0.15	0.07	0.05	0.13	0.01	0.12	0.20	0.04	0.20	0.14	0.10	0.06	0.05	0.12	0.03	2.00	0.083
18	0.07	0.02	0.01	0.03	0.02	0.02	0.03	0.16	0.10	0.20	0.18	0.19	0.26	0.03	0.06	0.09	0.16	0.24	0.08	0.08	0.06	0.02	0.02	0.07	2.20	0.092
19	0.03	0.04	0.02	0.06	0.02	0.05	0.04	0.04	0.10	0.20	0.02	0.03	0.06	0.01	0.06	0.05	0.06	0.12	0.09	0.05	0.06	0.02	0.02	0.03	1.31	0.055
20	0.04	0.01	0.06	0.03	0.03	0.02	0.05	0.02	0.04	0.06	0.02	0.03	0.03	0.09	0.04	0.05	0.05	0.06	0.06	0.08	0.02	0.03	0.02	0.04	0.98	0.041
21	0.04	0.07	0.02	0.02	0.02	0.03	0.05	0.07	0.20	0.10	0.02	0.04	0.26	0.25	0.06	0.08	0.14	0.20	0.10	0.20	0.14	0.06	0.03	0.02	2.22	0.093
22	0.02	0.05	0.06	0.02	0.01	0.02	0.02	0.04	0.03	0.09	0.10	0.04	0.12	0.07	0.02	0.04									0.73	0.046
23																										
24											0.06	0.03	0.02	0.03	0.01	0.03	0.02	0.07	0.02	0.03	0.02	0.03	0.02	0.06	0.45	0.032
25	0.06	0.07	0.02	0.01	0.01	0.04	0.05	0.03	0.04	0.04	0.03	0.07	0.03	0.03	0.04	0.03	0.02	0.02	0.03	0.02	0.02	0.01	0.02	0.03	0.75	0.031
26	0.03	0.03	0.06	0.04	0.02	0.03	0.03	0.02	0.10	0.02	0.08														0.46	0.042
27																										
28																										
29																										
30																										
31																										
total	0.86	0.55	0.55	0.64	0.44	0.53	0.68	1.11	1.38	1.65	1.11	1.11	1.61	1.42	1.05	1.29	1.16	1.65	0.99	1.09	0.86	0.55	0.60	0.60	23.48	0.052
Mean	0.015	0.031	0.031	0.034	0.023	0.029	0.036	0.058	0.073	0.087	0.062	0.065	0.095	0.081	0.058	0.072	0.068	0.097	0.055	0.061	0.048	0.031	0.033	0.033	0.055	

(E. W.) 1942, Oct

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2													0.03	0.05	0.10	0.02	0.04	0.02	0.06	0.03	0.13	0.12	0.02	0.03	0.65	0.054
3	0.04	0.04	0.14	0.03	0.04	0.04	0.04	0.06	0.05	0.13	0.23														0.84	0.076
4														0.02	0.08	0.08	0.10	0.02	0.02	0.03	0.02	0.01	0.01	0.02	0.41	0.037
5	0.02															0.07		0.04	0.01	0.01	0.02	0.02	0.01	0.03	0.23	0.026
6	0.01	0.02	0.02	0.01	0.01	0.01	0.02	0.01	0.10	0.06	0.09	0.10	0.06	0.07	0.04	0.03	0.02	0.04	0.02	0.02	0.02	0.01		0.04	0.83	0.036
7	0.02	0.01	0.02	0.02	0.05	0.02	0.02	0.04	0.02	0.01	0.02			0.03	0.05	0.04	0.02	0.02	0.02	0.03	0.02	0.02	0.01	0.02	0.56	0.025
8	0.02	0.01	0.01	0.02	0.01	0.02	0.02	0.03	0.02	0.02	0.02	0.02	0.03	0.03	0.04	0.03	0.02	0.02	0.03	0.02	0.02	0.02	0.04	0.02	0.57	0.024
9	0.02	0.02	0.02	0.05	0.03	0.02	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.01	0.03	0.03	0.05	0.03	0.02	0.02	0.02	0.02	0.02	0.03	0.61	0.025
10	0.02	0.02	0.03	0.02	0.04	0.02	0.02	0.02	0.02	0.03	0.05	0.02	0.02	0.02	0.03	0.06	0.03	0.11	0.06		0.02	0.03	0.02	0.02	0.73	0.032
11	0.02	0.02	0.01	0.01	0.02	0.02	0.03	0.06	0.16	0.04	0.04	0.03	0.03	0.06	0.10	0.08	0.09	0.02	0.02	0.11	0.02	0.02	0.02	0.03	1.06	0.044
12	0.04	0.04	0.02	0.04	0.02	0.06	0.08	0.02	0.04	0.10	0.23	0.30	0.12	0.15	0.12	0.10	0.16	0.15	0.05	0.04	0.03	0.02	0.05	0.05	2.03	0.085
13	0.09	0.03	0.04	0.02	0.03	0.02	0.07	0.06	0.10	0.15	0.20	0.16	0.10	0.09	0.06	0.04	0.07	0.08	0.07	0.02	0.02	0.05	0.06	0.03	1.66	0.069
14	0.10	0.02	0.06	0.02	0.03	0.03	0.05	0.04	0.06	0.02	0.03					0.03	0.11	0.13	0.05	0.10	0.07	0.06	0.02	0.02	1.05	0.053
15	0.08	0.02		0.03									0.15	0.10	0.18	0.06	0.03	0.10	0.02	0.02	0.03	0.05	0.06	0.03	0.96	0.064
16	0.03	0.03	0.06	0.10	0.08	0.02	0.01	0.03	0.02	0.03	0.10	0.09	0.08	0.06											0.74	0.053
17																		0.06	0.03	0.04	0.05	0.03	0.04	0.03	0.28	0.040
18	0.02	0.06	0.04	0.03	0.10	0.08	0.04	0.10	0.13	0.16															0.76	0.076
19															0.10	0.12	0.09	0.07	0.10	0.12	0.06	0.05	0.02	0.02	0.75	0.075
20	0.10	0.03	0.04	0.04	0.03	0.02	0.01	0.02	0.07	0.08	0.10	0.08	0.15	0.03	0.10	0.06	0.10	0.20	0.03	0.05	0.06	0.10	0.04	0.10	1.64	0.068
21	0.10	0.03	0.05	0.04	0.05	0.03	0.04	0.02	0.03	0.03	0.10	0.05	0.05	0.06	0.04	0.03	0.05	0.04	0.05	0.03	0.03	0.04	0.10	0.06	1.15	0.048
22	0.06	0.03	0.03	0.02	0.03	0.04	0.03	0.04	0.03	0.05	0.10	0.04	0.04	0.03	0.08	0.04	0.05	0.07	0.10	0.04	0.03	0.06	0.06	0.04	1.14	0.048
23	0.03	0.04	0.03	0.04	0.04	0.02	0.03	0.02	0.02	0.03	0.02	0.02	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.02	0.03	0.02	0.03	0.62	0.026
24	0.02	0.03	0.04	0.03	0.03	0.02	0.03	0.02	0.04	0.03	0.04	0.02	0.02	0.02	0.03	0.04	0.05	0.03	0.02	0.02	0.02	0.02	0.01	0.01	0.64	0.027
25	0.01	0.02	0.02	0.01	0.01	0.02																		0.02	0.11	0.016
26	0.02	0.02	0.02	0.02	0.04	0.02	0.02	0.03	0.06	0.10	0.11	0.09	0.04	0.05	0.07	0.06	0.04	0.03	0.02	0.03	0.02	0.01	0.14	0.13	1.18	0.049
27	0.02	0.02	0.03	0.03	0.03	0.04	0.03	0.07	0.10	0.19	0.08	0.06	0.06	0.03	0.04	0.03	0.06	0.04	0.03	0.02	0.03	0.03	0.02	0.10	1.19	0.050
28	0.10	0.04	0.03	0.02	0.02	0.02	0.02	0.06	0.01	0.10	0.10	0.03	0.03	0.04	0.03	0.02	0.01	0.02	0.01	0.02	0.02	0.01	0.15	0.05	0.99	0.041
29	0.02	0.10	0.02	0.02	0.02	0.04	0.10	0.06	0.03	0.04	0.05	0.03	0.10	0.10	0.09	0.08	0.07	0.06	0.02	0.03	0.02	0.01	0.02	0.01	1.14	0.048
30	0.02	0.01	0.02	0.06	0.05	0.07	0.05	0.19	0.17	0.09	0.08	0.06	0.09	0.10	0.26	0.06	0.17	0.07	0.03	0.10	0.07	0.02	0.03	0.02	1.89	0.079
31	0.02	0.02	0.05	0.02	0.03	0.10	0.20	0.06	0.09	0.10	0.26	0.14	0.02	0.13	0.10	0.06	0.07	0.03	0.08	0.02	0.03	0.03	0.02	0.20	1.88	0.078
total	1.05	0.76	0.80	0.78	0.82	0.80	0.99	1.08	1.42	1.65	2.08	1.36	1.27	1.30	1.79	1.30	1.53	1.51	0.99	1.00	0.90	0.89	1.01	1.19	28.29	0.049
Mean	0.040	0.030	0.035	0.050	0.033	0.033	0.043	0.047	0.062	0.072	0.095	0.072	0.060	0.057	0.078	0.052	0.064	0.058	0.038	0.040	0.035	0.034	0.040	0.044	0.050	

(E. W.) 1942, Nov

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.03	0.02	0.02	0.04	0.03						0.10	0.10	0.01	0.01	0.02	0.03	0.06		0.06	0.02	0.05	0.04	0.03	0.02	0.75	0.042
2	0.05	0.08	0.02	0.02	0.02	0.01						0.04	0.08		0.05	0.04	0.06		0.02				0.03		0.52	0.040
3											0.02	0.02	0.04	0.02	0.02	0.01	0.02	0.04	0.02	0.02	0.02		0.02	0.10	0.37	0.028
4	0.02	0.05	0.03	0.02	0.02	0.03	0.05	0.04					0.05	0.04	0.03	0.02		0.02			0.02	0.02	0.01	0.01	0.48	0.028
5	0.02	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.04	0.03	0.03													0.29	0.024
6															0.02	0.02	0.05	0.02	0.02	0.03	0.04	0.03	0.02	0.04	0.29	0.029
7															0.06	0.03	0.03	0.03	0.02	0.02	0.09	0.05	0.03	0.02	0.38	0.038
8	0.03	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.04	0.10	0.09	0.06	0.05	0.06	0.06	0.02	0.02	0.04	0.03	0.06	0.03	0.02	0.01	0.01	0.91	0.038
9	0.02	0.03					0.03	0.02	0.04	0.10	0.25	0.29	0.09	0.04	0.03	0.04									0.98	0.082
10														0.06	0.06	0.10	0.04	0.03	0.02	0.04	0.06	0.03	0.02	0.02	0.48	0.044
11	0.03	0.10	0.02	0.03	0.03	0.09	0.07	0.11					0.05	0.12	0.06	0.17	0.07	0.04	0.03	0.03	0.03	0.04	0.04	0.01	1.13	0.057
12	0.01	0.01	0.01	0.07	0.06	0.02	0.04	0.05	0.02	0.10	0.30		0.10	0.09	0.03	0.01	0.12	0.06	0.03	0.06	0.05	0.04	0.02	0.12	1.33	0.058
13	0.02	0.03	0.02	0.04	0.02	0.04	0.02	0.02	0.03	0.10	0.11	0.06	0.07	0.16	0.04										0.78	0.052
14													0.03	0.04	0.03	0.06	0.07	0.02	0.01	0.01	0.04	0.03	0.02	0.01	0.37	0.031
15	0.01	0.01	0.02	0.03	0.01	0.06	0.03	0.19	0.03	0.10	0.17	0.03	0.15	0.09	0.03	0.02	0.02	0.02	0.02	0.01	0.03	0.04	0.02	0.09	1.21	0.050
16	0.05	0.04	0.05	0.07	0.01	0.13	0.11	0.06	0.06	0.07	0.02	0.02	0.02	0.04	0.13	0.05	0.03	0.03	0.01	0.07	0.03	0.02	0.02	0.01	1.23	0.051
17	0.02	0.02	0.03	0.01	0.03	0.02			0.03	0.02	0.03	0.05	0.01	0.06	0.03	0.07	0.01	0.01	0.01	0.02	0.03	0.01	0.02	0.02	0.65	0.030
18	0.03	0.03	0.01	0.02	0.02	0.02	0.01	0.02	0.03	0.02	0.04	0.05	0.03	0.06	0.05	0.02		0.03	0.01	0.01	0.01	0.04	0.02	0.03	0.60	0.026
19	0.02	0.02	0.02	0.02	0.06	0.01	0.05	0.03	0.03	0.07	0.09	0.08													0.50	0.042
20															0.05	0.03	0.09	0.02	0.02	0.03	0.03	0.02	0.02	0.04	0.35	0.035
21	0.03	0.03	0.02	0.05	0.06	0.02	0.02	0.02	0.06	0.16	0.03	0.10	0.09	0.06	0.03	0.03	0.01	0.02	0.02	0.03	0.04	0.02	0.02	0.01	0.91	0.038
22	0.01	0.01	0.02	0.02	0.02	0.03	0.02	0.03	0.03	0.08	0.08	0.13	0.02	0.02	0.04	0.02	0.04								0.62	0.036
23																										
24													0.04	0.03	0.04	0.03	0.06	0.04	0.02	0.01					0.27	0.034
25				0.03	0.02								0.06	0.03	0.08	0.03	0.13	0.04	0.03	0.02					0.47	0.047
26																										
27																										
28																	0.03	0.01	0.07	0.04	0.04	0.05	0.02	0.03	0.29	0.036
29	0.01	0.06	0.03	0.03	0.02	0.06	0.02	0.02	0.02	0.03	0.02	0.02	0.02	0.03	0.02	0.01				0.06	0.03	0.03	0.02	0.02	0.58	0.028
30	0.06	0.02	0.03	0.02	0.02	0.03	0.03	0.02	0.06	0.10	0.03	0.06	0.07	0.02	0.03	0.05	0.03	0.02	0.01	0.02	0.02	0.03	0.02	0.02	0.88	0.037
31																										
Total	0.36	0.61	0.42	0.56	0.56	0.64	0.55	0.59	0.50	1.09	1.36	1.15	1.11	1.05	1.01	0.95	0.95	0.57	0.51	0.64	0.69	0.59	0.43	0.63	17.62	0.040
Mean	0.026	0.02	0.025	0.031	0.31	0.03	0.03	0.057	0.053	0.036	0.078	0.086	0.07	0.057	0.05	0.042	0.04	0.04	0.05	0.06	0.026	0.062	0.036	0.033	0.022	0.033	0.041	

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.02	0.03	0.08	0.03	0.06	0.02	0.02	0.03	0.03	0.03	0.02	0.06	0.03		0.02	0.02	0.03	0.02	0.01	0.02	0.02	0.02	0.03	0.05	0.70	0.030
2	0.02	0.02	0.01	0.04	0.02	0.03	0.02	0.03	0.04	0.06	0.02	0.05	0.02	0.04	0.05	0.03	0.01	0.03	0.03	0.02	0.02	0.03	0.03	0.08	0.75	0.031
3	0.06	0.02	0.04	0.03	0.03	0.04	0.02	0.02	0.05	0.08	0.04	0.03	0.05	0.02			0.02	0.02	0.02	0.04	0.01	0.02			0.66	0.033
4			0.02	0.02	0.05	0.02	0.03	0.02	0.01			0.02	0.03	0.02	0.02	0.04	0.02	0.02	0.03	0.02	0.05	0.02	0.02	0.04	0.52	0.026
5	0.02	0.02	0.02	0.03	0.03	0.02	0.04	0.04	0.04	0.02	0.09	0.03	0.03	0.02	0.04	0.04	0.02	0.03	0.02	0.02	0.08	0.06	0.04	0.06	0.86	0.036
6	0.10	0.06	0.03	0.02	0.03	0.03	0.02	0.03	0.02	0.02	0.04	0.03	0.02	0.02	0.02	0.04	0.04	0.02	0.06	0.02	0.03	0.03	0.03	0.02	0.78	0.033
7	0.02	0.03	0.04	0.10	0.04	0.02	0.02	0.03	0.02	0.04	0.05	0.06	0.04	0.03	0.02	0.03	0.03	0.02	0.02	0.03	0.02	0.05	0.03	0.04	0.83	0.035
8	0.03	0.04	0.03	0.04	0.03	0.03	0.04	0.03	0.11	0.02	0.04	0.02	0.04	0.02	0.02	0.03	0.02	0.03	0.01	0.02	0.02	0.02	0.03	0.03	0.75	0.031
9	0.04	0.02	0.04	0.02	0.02	0.04	0.03	0.02	0.03	0.08	0.08	0.04	0.04	0.07											0.57	0.041
10								0.16	0.08	0.05	0.02		0.04	0.13	0.05	0.09	0.06	0.03	0.02	0.03	0.02	0.02	0.03	0.02	0.85	0.053
11	0.01								0.06	0.05	0.02	0.06													0.20	0.040
12														0.08	0.12	0.09	0.06	0.05	0.04	0.05	0.06	0.04	0.04	0.05	0.68	0.062
13																0.04	0.05	0.02	0.04	0.04	0.02	0.01	0.02	0.01	0.25	0.028
14	0.01	0.02	0.02	0.01	0.03	0.02	0.03	0.01	0.04	0.06	0.01	0.01	0.03	0.03	0.04	0.03	0.02	0.04	0.02	0.01	0.02	0.02	0.02	0.02	0.57	0.024
15	0.03	0.04	0.06	0.03	0.02	0.02	0.02	0.01	0.02	0.09	0.02	0.06	0.03	0.04	0.06	0.01	0.04	0.01	0.02	0.01	0.02	0.04	0.02	0.02	0.74	0.031
16	0.04	0.04	0.10	0.02	0.05	0.02	0.03	0.02	0.06	0.06	0.08	0.04	0.02	0.06	0.02	0.02	0.03	0.01	0.02	0.02	0.02	0.04	0.02	0.02	0.86	0.035
17	0.01	0.02	0.02	0.04	0.04	0.06	0.03	0.02	0.02	0.03	0.01	0.03	0.05	0.03	0.02	0.03	0.02	0.06	0.03	0.02	0.02	0.03	0.02	0.04	0.70	0.029
18	0.02	0.02	0.02	0.02	0.04	0.04	0.04	0.02	0.04	0.05	0.05	0.06	0.04	0.04	0.02	0.06	0.06	0.02	0.03	0.02	0.04	0.03	0.02	0.02	0.81	0.033
19	0.04	0.02	0.03	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.04	0.05	0.04	0.02	0.02	0.02	0.01	0.01	0.02	0.02	0.03	0.02	0.02	0.04	0.61	0.025
20	0.04	0.01	0.02	0.01	0.02	0.01	0.02	0.02	0.03	0.03	0.02	0.03	0.14	0.02	0.04	0.02	0.02	0.02	0.02	0.01	0.01	0.02	0.03	0.02	0.66	0.028
21	0.02	0.02	0.01	0.04											0.04										0.13	0.026
22															0.04	0.13	0.04	0.22							0.43	0.108
23																										
24													0.07	0.06	0.04	0.06	0.04	0.02	0.01	0.01	0.02	0.01	0.02	0.03	0.39	0.033
25	0.04	0.04	0.02	0.02	0.02	0.02	0.02	0.01	0.02	0.02	0.03	0.03	0.03	0.03	0.05	0.05	0.03	0.03	0.03	0.04	0.02	0.01	0.01	0.02	0.64	0.027
26	0.03	0.03	0.02	0.02	0.02	0.03	0.04	0.04	0.01	0.01	0.01	0.02	0.09	0.08	0.08	0.06	0.03	0.01	0.02	0.03	0.02	0.02	0.02	0.02	0.71	0.030
27	0.02	0.03	0.04	0.04	0.02	0.02	0.03	0.03	0.05	0.03	0.03	0.05	0.02	0.02	0.02	0.06	0.02	0.04	0.01	0.03	0.04	0.02	0.02	0.02	0.68	0.028
28	0.05	0.02	0.03	0.03	0.01	0.02	0.04	0.01	0.02	0.04	0.03	0.02	0.02	0.02	0.02	0.03	0.01	0.01	0.01	0.02	0.02	0.04	0.01	0.02	0.55	0.023
29	0.03	0.02	0.18	0.03	0.05	0.02	0.03	0.04	0.02	0.03	0.04	0.04													0.53	0.044
30																0.12	0.02	0.02	0.03	0.02	0.01	0.02	0.03	0.04	0.31	0.034
31	0.02	0.03	0.01									0.02	0.04	0.02	0.05	0.03	0.02	0.02	0.03	0.02	0.07	0.02	0.01	0.01	0.42	0.026
total	0.72	0.60	0.89	0.62	0.55	0.58	0.57	0.63	0.84	0.93	0.78	0.84	0.96	0.92	0.92	1.17	0.77	0.86	0.60	0.59	0.71	0.66	0.57	0.74	18.14	0.035
Mean	0.03	0.03	0.03	0.02	0.03	0.02	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.04	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.03	0.03	

Table 2. NW-Component in mv/100meters at Khailar Blanks are the disturbed.

(N. S.) 1942, Jan.

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2																										
3																										
4												0.06	0.07	0.13	0.16	0.10	0.05	0.06	0.06	0.10	0.04	0.05	0.06	0.12	1.06	0.082
5	0.05	0.06	0.03	0.02	0.08	0.15	0.25	0.08	0.10	0.16	0.20	0.13	0.08	0.02	0.08	0.06	0.10	0.10	0.04	0.08	0.18	0.06	0.05	0.07	2.23	0.093
6	0.10	0.05	0.05	0.06	0.10	0.05	0.05	0.07	0.08	0.15	0.05	0.03			0.07	0.11	0.04	0.05	0.03	0.05	0.15	0.08	0.04	0.07	1.53	0.070
7	0.05	0.04	0.10	0.16	0.10	0.07	0.03	0.06	0.04	0.07	0.05	0.08	0.08	0.10	0.07	0.05	0.12	0.07	0.09	0.06	0.05	0.08	0.06	0.05	1.73	0.072
8	0.07	0.05	0.05	0.05	0.08	0.07	0.05	0.04	0.06	0.10	0.10		0.03	0.02	0.08	0.07	0.04	0.06	0.05	0.05	0.02	0.06	0.05	0.06	1.31	0.057
9	0.05	0.04	0.04	0.05	0.03	0.06	0.05	0.05	0.09	0.09			0.05	0.06	0.07	0.06	0.05	0.07	0.05	0.06	0.06	0.09	0.09	0.04	1.30	0.059
10	0.10	0.04	0.06	0.08	0.17	0.13	0.09																		0.67	0.096
11																										
12																	0.09	0.09	0.08	0.13	0.08	0.06	0.06	0.05	0.64	0.080
13	0.07	0.10	0.07	0.03	0.12	0.08	0.13	0.09	0.10	0.10															0.89	0.089
14																			0.05	0.07	0.05	0.05	0.02	0.05	0.29	0.048
15	0.05	0.03	0.02	0.03	0.05	0.03	0.07	0.02	0.06		0.03	0.04	0.05	0.05	0.05	0.03	0.05	0.06	0.04	0.02	0.07	0.09	0.08	0.10	1.12	0.049
16	0.10	0.05	0.05	0.06	0.03	0.06	0.03	0.03				0.02	0.03	0.03		0.03	0.06	0.08	0.06	0.07	0.09	0.10	0.02	0.03	1.03	0.052
17	0.03	0.02	0.03	0.02	0.07	0.03	0.06	0.05	0.03	0.04	0.04														0.42	0.038
18																										
19																										
20																										
21																										
22																	0.02	0.03	0.03	0.08	0.05	0.07	0.03	0.10	0.41	0.051
23	0.10	0.02	0.03	0.04	0.05	0.04	0.01	0.03	0.02	0.01	0.03	0.01	0.03	0.01	0.02	0.05	0.02	0.03	0.02	0.03	0.05	0.05	0.04	0.06	0.80	0.033
24	0.05	0.02	0.02	0.02	0.06	0.08	0.06	0.06	0.02	0.03															0.42	0.042
25																										
26												0.06	0.05	0.06	0.05	0.08	0.07	0.03	0.04	0.04	0.04	0.03	0.03	0.04	0.62	0.048
27	0.04	0.02	0.05	0.03	0.03	0.05	0.04	0.02	0.05	0.05	0.10	0.09	0.07	0.07	0.07	0.04	0.06	0.05	0.09	0.03	0.05	0.03	0.05	0.04	1.22	0.051
28	0.04	0.05	0.05	0.03	0.05	0.03	0.03	0.05	0.03	0.06	0.08	0.06	0.04	0.03	0.05	0.03	0.04	0.05	0.03	0.04	0.03	0.01	0.10	0.04	1.05	0.044
29	0.05	0.05	0.05	0.05	0.03	0.08	0.05	0.03	0.02			0.05	0.04	0.10	0.05	0.04	0.07	0.04	0.05	0.02	0.03	0.04	0.08	0.05	1.07	0.049
30	0.03	0.03	0.04	0.06	0.04	0.07	0.09	0.07	0.03	0.15	0.12	0.07	0.04	0.06	0.05	0.02	0.05	0.02	0.05	0.04	0.03	0.05	0.07	0.07	1.35	0.056
31	0.05	0.04	0.02	0.03	0.07	0.03	0.05	0.04	0.05	0.06	0.05														0.49	0.045
total	1.03	0.71	0.76	0.82	1.16	1.11	1.14	0.79	0.78	1.07	0.85	0.70	0.66	0.74	0.87	0.77	0.93	0.89	0.86	0.97	1.07	1.00	0.93	1.04	21.65	0.059
Mean	0.061	0.042	0.045	0.048	0.068	0.65	0.067	0.049	0.052	0.082	0.077	0.058	0.051	0.057	0.067	0.055	0.058	0.056	0.051	0.057	0.063	0.059	0.055	0.061	0.059	

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.03	0.03	0.05	0.02	0.02	0.02	0.03	0.02	0.03	0.02	0.03	0.03	0.05	0.05	0.04	0.05	0.03	0.03	0.03	0.02	0.03	0.03	0.04	0.05	0.78	0.033
2	0.05	0.05	0.06	0.06	0.03	0.02	0.04	0.15	0.09	0.05	0.06	0.06	0.05	0.04	0.03	0.03	0.05	0.04	0.10	0.02	0.04	0.04	0.02	0.03	1.21	0.050
3	0.02	0.05	0.05	0.05	0.04	0.03	0.03	0.02	0.08	0.08	0.11	0.09	0.06	0.02	0.02	0.03	0.03	0.05	0.02	0.03	0.01	0.02	0.03	0.05	1.02	0.043
4	0.05	0.03	0.02	0.04	0.03	0.02	0.08	0.01	0.02	0.03	0.03	0.05	0.08	0.08	0.05	0.06	0.05	0.02	0.03	0.02	0.03	0.05	0.07	0.06	1.01	0.042
5	0.03	0.03	0.02	0.02	0.03	0.05	0.02	0.08	0.09	0.07	0.02	0.05	0.05	0.08	0.08	0.07	0.05	0.05	0.06	0.02	0.05	0.02	0.04	0.04	1.12	0.047
6	0.03	0.04	0.03	0.05	0.06	0.03	0.09	0.09	0.07	0.05	0.07	0.05	0.08	0.08	0.12	0.12	0.05	0.05	0.03	0.10	0.07	0.05	0.10	0.05	1.56	0.065
7	0.05	0.05	0.04	0.08	0.08	0.10	0.03	0.10	0.15	0.08	0.07	0.12	0.10	0.08	0.06	0.07	0.06	0.06	0.07	0.03	0.04	0.06	0.05	0.02	1.65	0.069
8	0.05	0.02	0.05	0.02	0.03																				0.17	0.034
9																										
10																										
11																										
12																										
13																0.05	0.07	0.06	0.04	0.04	0.03	0.08	0.05	0.08	0.50	0.056
14	0.02	0.03	0.03	0.04	0.04	0.07	0.05	0.05	0.05	0.07	0.07	0.06	0.05	0.03	0.10	0.10	0.03	0.05	0.06	0.05	0.04	0.03	0.05	0.10	1.27	0.053
15	0.07	0.02	0.03	0.02	0.05	0.05	0.03	0.04	0.09	0.10	0.04	0.08	0.05	0.03	0.05	0.09	0.05	0.04	0.06	0.07	0.10	0.03	0.02	0.04	1.25	0.053
16	0.03	0.02	0.11	0.10	0.05	0.06	0.07	0.04	0.10	0.05	0.15	0.08	0.06	0.05	0.10	0.13	0.10	0.10	0.13	0.09	0.08	0.06	0.04	0.09	1.89	0.079
17	0.05	0.02	0.02	0.02	0.08	0.05	0.08	0.10	0.08	0.10	0.12	0.12	0.10	0.12	0.05	0.08	0.03	0.04	0.03	0.04	0.02	0.04	0.03	0.01	1.43	0.060
18	0.02	0.01	0.02	0.01	0.06	0.09	0.08	0.02	0.04	0.12	0.12	0.15	0.13	0.10	0.10	0.08	0.03	0.05	0.05	0.05	0.07	0.02	0.05	0.01	1.48	0.062
19	0.02	0.02	0.02	0.01	0.06	0.03	0.05	0.04	0.04	0.02	0.03	0.10	0.02	0.04	0.10	0.15	0.05	0.02	0.01	0.02	0.01	0.03	0.02	0.03	0.94	0.039
20	0.10	0.02	0.03	0.03	0.05	0.03	0.05	0.07	0.06	0.10	0.10	0.12	0.10	0.05	0.03	0.10	0.05	0.07	0.04	0.09	0.08	0.04	0.06	0.03	1.50	0.063
21	0.05	0.02	0.04	0.03	0.03	0.04	0.06	0.06	0.03	0.02	0.06	0.10	0.10		0.05	0.15	0.09	0.05	0.03	0.03	0.05	0.07	0.04	0.11	1.31	0.057
22	0.05	0.04	0.15	0.07	0.10	0.03	0.04	0.03	0.05	0.10	0.06	0.05	0.02	0.07	0.02	0.02	0.01	0.05	0.03	0.01	0.02	0.03	0.03	0.02	1.10	0.046
23	0.07	0.06	0.06	0.10	0.05	0.06	0.04	0.15				0.10	0.09	0.04	0.05	0.03	0.05	0.03	0.02	0.04	0.02	0.02	0.04	0.10	1.22	0.058
24	0.17	0.18	0.13	0.12	0.22	0.04	0.17	0.13	0.20	0.23			0.10	0.10	0.12	0.09	0.08	0.09	0.08	0.12	0.10	0.05	0.09	0.10	2.71	0.123
25	0.05	0.08	0.03	0.07	0.04	0.05	0.07	0.06	0.06	0.12	0.11	0.11	0.12	0.20	0.05	0.08	0.08	0.08	0.05	0.05	0.03	0.03	0.06	0.06	1.74	0.073
26	0.06	0.04	0.10	0.09	0.05	0.12	0.13	0.10	0.05	0.25		0.24	0.22	0.16	0.14	0.11	0.09	0.10	0.05	0.06	0.07	0.06	0.05	0.08	2.42	0.105
27	0.06	0.06	0.08	0.06	0.05	0.06	0.05	0.05	0.05	0.06	0.07						0.02	0.01	0.02	0.02	0.03	0.04	0.03	0.03	0.85	0.045
28	0.02	0.06	0.09	0.06	0.10	0.05	0.11	0.09	0.12	0.11	0.08	0.05	0.06	0.04											1.04	0.074
29																										
30																										
31																										
over	1.15	0.98	1.26	1.17	1.35	1.10	1.40	1.50	1.55	1.83	1.40	1.81	1.69	1.46	1.36	1.69	1.15	1.14	1.04	1.02	1.02	0.90	1.01	1.19	31.17	0.060
Mean	0.050	0.043	0.055	0.051	0.059	0.050	0.064	0.068	0.074	0.087	0.074	0.091	0.080	0.073	0.068	0.080	0.052	0.052	0.047	0.046	0.046	0.041	0.046	0.054	0.060	

(N. S.) 1942, March.

day \ hour	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2											0.30	0.35	0.10	0.07	0.03	0.02	0.04	0.03	0.03	0.04	0.05	0.15	0.08	0.05	1.34	0.096
3	0.04	0.05	0.06	0.05	0.05	0.06	0.07	0.05	0.07	0.08	0.07	0.08	0.06	0.06	0.06	0.12	0.12	0.05	0.16	0.08	0.13	0.10	0.06	0.20	1.93	0.080
4	0.20	0.08	0.07	0.03	0.05	0.08	0.07					0.15	0.10	0.05	0.05	0.07	0.17	0.12	0.06	0.16	0.07	0.10	0.08	0.10	1.84	0.092
5	0.08	0.12	0.09	0.13	0.10	0.08	0.09	0.08	0.08		0.10	0.10	0.09	0.13	0.10	0.13	0.08	0.05	0.04	0.04	0.05	0.07	0.08	0.05	1.96	0.085
6	0.15	0.15	0.15	0.15	0.30	0.10	0.23	0.10	0.05	0.10	0.10	0.14	0.11	0.05	0.05	0.10	0.03	0.05	0.05	0.12	0.05	0.03	0.08	0.04	2.46	0.103
7	0.18	0.02	0.02	0.03	0.03	0.01	0.01	0.05	0.09	0.13	0.10	0.13	0.05	0.04	0.06	0.04	0.09	0.09	0.13	0.07	0.08	0.10	0.06	0.03	1.74	0.073
8	0.15	0.05	0.01	0.05	0.10	0.08	0.15	0.10	0.10	0.30	0.10	0.13	0.10	0.12	0.09	0.10	0.12	0.12	0.08	0.06	0.10	0.10	0.03	0.05	2.30	0.096
9	0.10	0.13	0.08	0.09	0.08	0.10	0.09	0.07	0.08	0.10	0.14	0.10	0.15	0.13	0.30	0.10	0.04	0.05	0.05	0.08	0.09	0.12	0.10	0.12	2.49	0.104
10	0.09	0.01	0.02	0.06	0.10	0.07	0.06	0.04	0.05	0.07	0.05	0.04	0.08	0.04	0.10	0.04	0.08	0.17	0.05	0.04	0.06	0.05	0.08	0.05	1.53	0.064
11	0.05	0.04	0.07	0.05	0.08	0.05	0.07	0.07	0.05	0.06	0.20	0.08	0.16	0.07	0.05	0.05	0.12	0.12	0.10	0.11	0.05	0.08	0.05	0.03	1.86	0.078
12	0.05	0.04	0.04	0.05	0.08	0.10	0.10	0.05	0.10	0.14	0.13	0.20	0.10	0.10		0.10	0.09	0.09	0.10	0.09	0.10	0.12	0.05	0.05	2.07	0.090
13	0.06	0.05	0.03	0.04	0.06	0.05	0.07	0.07	0.10	0.15	0.15	0.19	0.14	0.10	0.10	0.08	0.07	0.10	0.09	0.10	0.08	0.08	0.12	0.14	2.26	0.094
14	0.07	0.06	0.03	0.08	0.09	0.09	0.07	0.11	0.12	0.12	0.10	0.10	0.09	0.09		0.04	0.06	0.05	0.04	0.04	0.06	0.03	0.02	0.04	1.60	0.070
15	0.03	0.04	0.04	0.06	0.06	0.04	0.05	0.05	0.06	0.05	0.03	0.04	0.04	0.05	0.06	0.08	0.09	0.08	0.10	0.08	0.06	0.04	0.02	0.05	1.33	0.055
16	0.05	0.08	0.08	0.07	0.05	0.06	0.10	0.15	0.15	0.12	0.07	0.06	0.08	0.07	0.05	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.03	0.02	1.42	0.059
17	0.02	0.01	0.01	0.02	0.02	0.02	0.05	0.05	0.05	0.05	0.10	0.10	0.05	0.05	0.05	0.03	0.06	0.09	0.04	0.05	0.03	0.02	0.02	0.02	1.04	0.043
18	0.02	0.02	0.02	0.04	0.04	0.03	0.07	0.07	0.07	0.05	0.08	0.06	0.05	0.03	0.04	0.05	0.04	0.03	0.04	0.04	0.06	0.04	0.07	0.10	1.16	0.048
19	0.02	0.08	0.06	0.03	0.05	0.08	0.07	0.05	0.05	0.07	0.06	0.06	0.05	0.07	0.05	0.06	0.06	0.08	0.07	0.08	0.03	0.05	0.07	0.05	1.41	0.059
20	0.04	0.04	0.07	0.09	0.04	0.07	0.05	0.07	0.05	0.06	0.07	0.06	0.10	0.03	0.05	0.10	0.05	0.07	0.09	0.04	0.07	0.07	0.07	0.04	1.49	0.062
21	0.04	0.07	0.05	0.03	0.05	0.06	0.03	0.04	0.03																0.40	0.044
22																										
23																										
24												0.10	0.08	0.10	0.10	0.07	0.10	0.10	0.12	0.12	0.08	0.08	0.09	0.07	1.21	0.093
25	0.09	0.08	0.07	0.02	0.05	0.03	0.03	0.05	0.07	0.06	0.10	0.11	0.06	0.08	0.05	0.08	0.06	0.03	0.08	0.07	0.05	0.07	0.03	0.05	1.52	0.063
26	0.04	0.05	0.05	0.04	0.05	0.03	0.04	0.05	0.07	0.05	0.09	0.06	0.10	0.10	0.04	0.05	0.06	0.10	0.12	0.11	0.10	0.13	0.12	0.16	1.81	0.075
27	0.10	0.07	0.13	0.37	0.08	0.10	0.04	0.15	0.04	0.05	0.15	0.03	0.10	0.10	0.04	0.03	0.06	0.08	0.05	0.04	0.03	0.04	0.02	0.05	1.95	0.081
28	0.08	0.04	0.03	0.05	0.12	0.06	0.08	0.09	0.10	0.07	0.08	0.04	0.05	0.06	0.07	0.04	0.04	0.06	0.05	0.04	0.04	0.05	0.05	0.03	1.42	0.059
29	0.03	0.05	0.05	0.04	0.04	0.03	0.04	0.03	0.05	0.06	0.05	0.07	0.06	0.05	0.03	0.04	0.07	0.05	0.03	0.09	0.05	0.04	0.04	0.06	1.15	0.048
30	0.01	0.06	0.05	0.06	0.06	0.10	0.06	0.07	0.07	0.04	0.06	0.07	0.10	0.12	0.10	0.04									1.10	0.069
31																										
total	1.72	1.53	1.49	1.73	1.83	1.67	1.84	1.66	1.76	2.02	2.48	2.65	2.25	1.96	1.72	1.69	1.83	1.88	1.79	1.82	1.60	1.78	1.52	1.65	43.78	0.073
Mean	0.068	0.061	0.056	0.069	0.073	0.067	0.074	0.069	0.073	0.092	0.103	0.102	0.087	0.075	0.072	0.065	0.073	0.075	0.072	0.073	0.064	0.071	0.061	0.066	0.073	

Date / Hour	9	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2																		0.05	0.06	0.06	0.07		0.08	0.10	0.42	0.070
3	0.18	0.04	0.06	0.07	0.07	0.16	0.18	0.08	0.06	0.07	0.19	0.10	0.10	0.10	0.07	0.09	0.08	0.07	0.15	0.08	0.03	0.05	0.06	0.06	2.11	0.087
4	0.12	0.06	0.05	0.04	0.05	0.05	0.07	0.05	0.05	0.07	0.01						0.15	0.10	0.09	0.09	0.10	0.18	0.12	0.09	1.60	0.084
5	0.12	0.11	0.10	0.30	0.32	0.22	0.05	0.10	0.12	0.09	0.09	0.07	0.06	0.06	0.01	0.05	0.06	0.05	0.07	0.10	0.13	0.10	0.09	0.12	2.62	0.109
6	0.06	0.06	0.06	0.05	0.06	0.14	0.05	0.06	0.06	0.09	0.08	0.05	0.07	0.08	0.08	0.07	0.05	0.07	0.05	0.05			0.05	0.07	1.53	0.060
7	0.02											0.05	0.07	0.03			0.07	0.03	0.03	0.03	0.05	0.01	0.04	0.03	0.53	0.041
8	0.08	0.05	0.02	0.03	0.05	0.04	0.06	0.05	0.09	0.04	0.07	0.08	0.06	0.07	0.07	0.05	0.07	0.16	0.10	0.07	0.10	0.09	0.01	0.08	1.76	0.073
9	0.10	0.12	0.10	0.06	0.05	0.10	0.12	0.08	0.10	0.15	0.20	0.12	0.06	0.05	0.07	0.09	0.09	0.10	0.03	0.01	0.06	0.09	0.07	0.12	2.30	0.095
10	0.10	0.07	0.07	0.06	0.06	0.12	0.06	0.12	0.05	0.05	0.01	0.07	0.06	0.10	0.05	0.01	0.01	0.03	0.05	0.06	0.05	0.07	0.02	0.01	1.46	0.061
11	0.05	0.04	0.03	0.03	0.05	0.05	0.09	0.07	0.12	0.13	0.10	0.08	0.05		0.10	0.20	0.15	0.09	0.12	0.10	0.05	0.05	0.08	0.01	1.94	0.081
12	0.07	0.03	0.03	0.03	0.03	0.07	0.05	0.06	0.05	0.10	0.06	0.05	0.06	0.04	0.04	0.03	0.07	0.05	0.05	0.03	0.04	0.02	0.06	0.02	1.16	0.048
13	0.03	0.05	0.02	0.02	0.03	0.03	0.06	0.05	0.10	0.09	0.05	0.04	0.02	0.08	0.06	0.03	0.07	0.03	0.10	0.09	0.03	0.01	0.05	0.10	1.35	0.057
14	0.06	0.09	0.04	0.04	0.05	0.06	0.07	0.09	0.12	0.15	0.17	0.13	0.12	0.05	0.10	0.07	0.15	0.10	0.06	0.10	0.10	0.06	0.07	0.04	2.26	0.094
15	0.04	0.03	0.02	0.05	0.06	0.04	0.05	0.02	0.05	0.05	0.05	0.07	0.05	0.09	0.05	0.03	0.03	0.01	0.06	0.03	0.05	0.01	0.06	0.01	1.16	0.048
16	0.03	0.02	0.02	0.03	0.06	0.05	0.05	0.08	0.08	0.06	0.02	0.03	0.05	0.05	0.05	0.05	0.06	0.03	0.05	0.07	0.03	0.01	0.05	0.04	1.28	0.056
17	0.03	0.04	0.05	0.04	0.06	0.12	0.10	0.06	0.10	0.06	0.06	0.15	0.18	0.17	0.10	0.07	0.12	0.13	0.10	0.06	0.08	0.04	0.03	0.02	1.93	0.084
18	0.08	0.07	0.03	0.04	0.02	0.05	0.05	0.03	0.06	0.06	0.10	0.07	0.10	0.09	0.02	0.05	0.06	0.10	0.05	0.07	0.05	0.03	0.01	0.03	1.40	0.058
19	0.06	0.03	0.06	0.04	0.10	0.08	0.06			0.08	0.10	0.10	0.08	0.05	0.07	0.07	0.08	0.08	0.06	0.05	0.04	0.06	0.06	0.08	1.42	0.065
20	0.03	0.02	0.01	0.03	0.03	0.05	0.05	0.05	0.05	0.05	0.10	0.08	0.07	0.06	0.05	0.05	0.07	0.07	0.03	0.02	0.04	0.03	0.01	0.04	1.04	0.043
21	0.04	0.04	0.04	0.04	0.04	0.05	0.04	0.04	0.04	0.06	0.06	0.04	0.08	0.06	0.04	0.04	0.05	0.04	0.03	0.03	0.04	0.04	0.05	0.03	1.08	0.045
22	0.02	0.02	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.05	0.03	0.03	0.03	0.03	0.04	0.05	0.05	0.03	0.03	0.03	0.02	0.01	0.01	0.80	0.033
23	0.05	0.06	0.06	0.05	0.05	0.04	0.04	0.05	0.05	0.05	0.05	0.04	0.05	0.09	0.06	0.10	0.07	0.05	0.05	0.05	0.02	0.07	0.04	0.04	1.55	0.065
24	0.03	0.03	0.03	0.03	0.05	0.06	0.05	0.10	0.13	0.10	0.12	0.12	0.10	0.05	0.05	0.04	0.07	0.07	0.06	0.09	0.05	0.04	0.07	0.04	1.78	0.074
25	0.04	0.05	0.03	0.03	0.05	0.07	0.04	0.02																	0.33	0.044
26																										
27														0.03	0.03	0.02	0.03	0.02	0.03	0.03	0.02				0.24	0.027
28																										
29																										
30														0.10	0.08	0.06	0.05				0.03	0.02	0.02	0.02	0.41	0.047
31																										
total	1.44	1.24	1.08	1.18	1.45	1.61	1.56	1.48	1.58	1.58	1.97	1.62	1.66	1.46	1.30	1.54	1.78	1.65	1.46	1.59	1.28	1.10	1.24	1.30	35.22	0.063
mean	0.048	0.041	0.036	0.039	0.048	0.054	0.052	0.049	0.053	0.053	0.066	0.054	0.055	0.049	0.043	0.051	0.059	0.055	0.049	0.053	0.043	0.037	0.041	0.043		0.056

(N. S.) 1942, May.

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1											0.05	0.10	0.05	0.02	0.06	0.10	0.11	0.03	0.01	0.02	0.02	0.02	0.03	0.02	0.64	0.046
2	0.01	0.02	0.05	0.04	0.02	0.03	0.01	0.02	0.05	0.20	0.22				0.02	0.03	0.02	0.05	0.03	0.01	0.02	0.03	0.03	0.02	0.93	0.044
3	0.08	0.01	0.02	0.05	0.01	0.02	0.04	0.03	0.01	0.02	0.05	0.10	0.06	0.03	0.05	0.04	0.04	0.05	0.05	0.06		0.04	0.02	0.02	0.90	0.039
4	0.01	0.04	0.03	0.02	0.02	0.02	0.02	0.05	0.08	0.02		0.03	0.03	0.05	0.04	0.04	0.05	0.04	0.02	0.04	0.05	0.03	0.03	0.03	0.79	0.034
5	0.02	0.04	0.07	0.08	0.05	0.01	0.02	0.05	0.04	0.05	0.10	0.03	0.07	0.02		0.04	0.03	0.05	0.06						0.83	0.046
6										0.09	0.08	0.05	0.05	0.02	0.03	0.02	0.01	0.05	0.01	0.02	0.05	0.05	0.04	0.02	0.59	0.039
7	0.03	0.01	0.04	0.01	0.02	0.01	0.02	0.01	0.01	0.04			0.05	0.04	0.01	0.01	0.05	0.01	0.01	0.02	0.02	0.03	0.03	0.02	0.50	0.023
8	0.01	0.02	0.04	0.05	0.03	0.04	0.05	0.03	0.03	0.02	0.03	0.05	0.04	0.07	0.05	0.09	0.10	0.06	0.04	0.02	0.02	0.02	0.03	0.02	0.96	0.040
9	0.04	0.08	0.03	0.04	0.03	0.02	0.02	0.08	0.02	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.03	0.02	0.01	0.01	0.02	0.02	0.01	0.61	0.025
10	0.01	0.02	0.01	0.01	0.03	0.05	0.02	0.01	0.03	0.02	0.02	0.02	0.05	0.01	0.02	0.01	0.02	0.02	0.01	0.01	0.02	0.05	0.05	0.03	0.55	0.023
11	0.05	0.04	0.08	0.04																					0.21	0.053
12																										
13																										
14																										
15												0.03	0.02	0.04	0.05	0.01	0.02	0.01	0.02	0.02	0.04	0.02	0.02	0.02	0.32	0.025
16	0.01	0.05	0.01	0.02	0.02	0.16	0.02	0.05	0.03	0.02	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.02	0.01	0.02	0.01	0.02	0.02	0.62	0.026
17	0.02	0.02	0.01	0.01	0.03	0.02	0.02	0.02	0.03	0.03	0.03	0.04	0.03	0.02	0.02	0.02	0.01	0.01	0.02	0.04	0.02	0.02	0.01	0.01	0.51	0.021
18	0.01	0.01	0.02	0.03	0.02	0.01	0.02	0.04	0.04	0.02	0.03														0.25	0.023
19											0.03	0.02	0.02	0.03	0.02	0.03	0.02	0.03	0.03	0.02	0.03	0.02	0.02	0.02	0.34	0.024
20	0.02	0.02	0.02	0.02	0.05	0.02	0.03	0.03	0.07	0.06	0.04	0.05	0.03	0.02	0.02	0.01	0.02	0.03	0.02	0.03	0.03	0.02	0.02	0.01	0.69	0.029
21	0.04	0.03	0.02	0.08	0.03	0.02	0.05	0.06	0.09	0.05	0.03	0.05	0.04	0.02	0.05	0.06	0.03	0.03	0.02	0.01	0.02	0.04	0.01	0.01	0.89	0.037
22	0.05	0.05	0.04	0.02	0.01	0.02	0.01	0.02	0.04	0.02	0.03	0.03	0.06	0.03	0.02	0.03	0.05	0.08	0.07	0.03	0.05	0.03	0.02	0.02	0.83	0.035
23	0.04	0.03										0.05	0.07	0.03	0.03	0.03	0.03	0.02	0.03	0.01	0.02	0.04	0.03	0.02	0.48	0.032
24	0.05	0.02	0.02	0.05	0.10	0.09	0.08	0.05	0.03	0.05	0.07	0.10	0.05	0.02	0.03	0.03	0.03	0.03	0.02	0.05	0.02	0.06	0.07	0.03	1.15	0.048
25	0.03	0.02	0.02	0.03	0.05	0.04	0.05	0.03	0.10		0.03	0.03	0.02	0.02	0.04	0.03	0.04	0.02	0.03	0.04	0.02	0.02	0.02	0.02	0.75	0.033
26															0.03	0.02	0.02	0.03	0.02	0.02	0.01	0.03	0.08	0.04	0.30	0.030
27	0.05	0.06	0.03	0.05	0.03	0.03	0.04	0.03	0.05	0.05	0.02		0.02	0.01	0.02	0.02	0.01	0.01	0.02	0.03	0.03	0.02	0.02	0.03	0.68	0.030
28	0.02	0.04	0.01	0.04	0.12	0.04	0.02	0.07	0.15	0.12	0.10	0.08	0.08	0.06	0.07	0.05	0.04	0.04	0.03	0.02	0.01	0.01	0.01	0.02	1.25	0.052
29	0.03	0.02	0.02	0.02	0.03	0.03	0.03	0.07	0.06		0.05	0.05	0.09	0.06	0.03	0.02	0.08	0.07	0.03	0.05	0.05	0.02	0.03	0.03	0.97	0.042
30	0.02	0.02	0.03	0.03	0.04	0.02	0.02	0.01	0.03	0.02	0.02	0.02													0.28	0.023
31																										
total	0.65	0.67	0.62	0.74	0.74	0.70	0.59	0.76	0.99	0.93	1.08	0.97	0.97	0.65	0.73	0.76	0.85	0.81	0.64	0.59	0.58	0.65	0.66	0.49	17.82	0.034
Mean	0.030	0.030	0.030	0.035	0.037	0.035	0.030	0.038	0.050	0.049	0.051	0.046	0.044	0.030	0.032	0.032	0.035	0.034	0.027	0.026	0.026	0.028	0.029	0.021	0.034	

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2											0.01	0.02	0.01	0.02	0.02	0.01	0.01	0.01	0.03	0.02	0.01	0.01	0.01	0.02	0.21	0.015
3	0.03	0.02	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.03	0.01	0.03	0.02	0.02	0.01	0.03	0.02	0.02	0.01	0.04	0.02	0.02	0.02	0.44	0.018
4	0.02	0.03	0.03	0.02	0.02	0.02	0.02	0.03																	0.19	0.024
5																										
6																	0.03	0.03	0.02	0.05	0.03	0.08	0.03	0.02	0.29	0.036
7	0.02	0.05	0.03	0.02	0.02	0.03	0.04	0.03	0.04	0.08	0.07	0.06	0.03	0.02	0.03	0.03	0.02	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.79	0.033
8	0.02	0.02	0.02	0.03	0.03	0.01	0.01	0.02	0.02	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.02	0.03	0.02	0.02	0.02	0.03	0.05	0.02	0.56	0.023
9	0.01	0.01	0.02	0.02	0.03	0.02	0.02	0.03	0.03	0.04	0.03	0.05	0.03	0.02	0.02	0.03	0.04	0.03	0.02	0.02	0.03	0.05	0.03	0.02	0.65	0.027
10	0.02	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.04	0.02	0.03	0.02		0.02	0.03	0.02	0.03	0.01	0.01	0.01	0.01	0.02	0.02	0.51	0.022
11	0.02	0.01	0.01	0.03	0.02	0.03	0.03	0.03	0.02																0.20	0.022
12											0.08	0.07	0.05	0.04	0.05	0.07	0.06	0.07	0.03	0.05	0.05	0.04	0.04	0.02	0.72	0.051
13	0.03	0.08	0.02	0.03	0.04	0.05	0.03	0.05	0.10	0.07	0.04	0.06	0.07	0.03	0.07	0.03	0.04	0.02	0.06	0.02	0.04	0.02	0.02	0.03	1.05	0.044
14	0.01	0.10	0.03	0.07	0.02	0.04	0.03	0.02	0.06	0.07	0.25	0.13	0.02	0.10	0.05	0.07	0.06	0.08	0.03	0.04	0.03	0.05	0.02	0.02	1.40	0.058
15	0.04	0.06	0.03	0.03	0.02	0.04	0.03	0.04	0.02	0.04	0.02	0.05	0.03	0.09	0.04	0.03	0.02	0.02	0.02	0.01	0.02	0.02	0.01	0.01	0.74	0.031
16	0.01	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.04	0.02	0.02	0.06	0.05	0.02	0.01	0.02	0.02	0.01	0.03	0.02	0.02	0.02	0.02	0.61	0.025
17	0.02	0.02	0.02	0.02	0.03	0.02	0.02	0.02	0.02	0.05	0.08	0.02	0.04	0.05	0.05	0.05	0.07	0.04	0.03	0.04	0.03	0.04	0.05	0.03	0.86	0.036
18	0.02	0.03	0.03	0.02	0.05	0.03	0.03	0.03	0.07	0.04	0.05	0.08	0.10	0.07	0.02	0.02	0.06	0.04	0.06	0.06	0.03	0.01	0.01	0.01	0.97	0.040
19	0.05	0.02	0.02	0.04	0.02	0.04	0.02	0.02	0.03	0.07	0.08	0.03	0.02	0.02	0.02	0.01	0.02	0.02	0.02	0.04	0.05	0.05	0.08	0.05	0.84	0.035
20	0.15	0.03	0.07	0.03	0.02	0.02	0.07	0.09	0.05	0.07	0.10	0.10	0.10	0.07	0.02	0.04	0.03	0.02	0.02	0.01	0.02	0.04	0.02	0.02	1.21	0.050
21	0.03	0.02	0.01	0.01	0.05	0.03	0.03	0.02	0.04	0.05	0.05	0.03	0.02	0.02	0.03	0.03	0.03	0.04	0.03	0.03	0.02	0.05	0.05	0.05	0.77	0.032
22	0.05	0.03	0.04	0.28	0.05	0.07																			0.52	0.087
23											0.03	0.02	0.03	0.04	0.05	0.05	0.04	0.05	0.09	0.05	0.02	0.05	0.03	0.07	0.62	0.044
24	0.09	0.04	0.35	0.04	0.04	0.03	0.03	0.10	0.10	0.11															0.93	0.093
25																										
26																										
27																										
28																										
29																										
30																										
31																										
total	0.64	0.61	0.78	0.76	0.53	0.55	0.47	0.60	0.67	0.81	0.98	0.80	0.71	0.69	0.55	0.54	0.62	0.60	0.54	0.53	0.49	0.61	0.53	0.47	15.08	0.038
Mean	0.036	0.034	0.043	0.042	0.029	0.031	0.028	0.035	0.042	0.054	0.058	0.047	0.042	0.043	0.032	0.032	0.034	0.033	0.030	0.029	0.027	0.034	0.029	0.026	0.036	

(N. S.) 1942. July.

day \ hour	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1										0.12	0.18	0.07	0.08	0.06	0.12	0.22	0.06	0.07	0.09	0.09	0.07	0.06	0.03	0.04	1.36	0.091
2	0.12	0.04	0.03	0.05	0.08	0.06	0.06	0.07	0.08	0.02	0.02	0.02	0.07	0.09	0.06	0.02	0.02	0.07	0.08	0.02	0.06	0.02	0.02	0.01	1.19	0.050
3	0.02	0.02	0.05	0.04	0.10	0.06	0.13	0.12	0.11	0.04	0.08	0.13	0.02	0.06	0.01	0.09	0.09	0.05	0.02	0.02	0.02	0.12	0.03	0.02	1.45	0.060
4	0.05	0.02	0.02	0.03	0.02	0.01	0.02	0.06	0.05	0.08	0.12	0.12	0.07	0.08	0.10	0.06	0.06	0.08	0.09						1.14	0.060
5											0.17	0.10	0.07	0.03	0.03	0.08	0.07	0.06	0.03	0.03	0.03	0.03	0.03	0.03	0.79	0.056
6	0.06	0.02	0.02	0.02	0.03	0.07	0.05						0.07	0.09	0.11	0.10	0.03	0.09	0.07	0.06	0.05	0.07	0.06	0.02	1.09	0.057
7	0.02	0.04	0.03	0.05	0.09	0.06	0.02	0.10	0.17	0.10	0.15	0.09	0.05	0.12	0.06	0.04	0.03	0.06	0.04	0.02		0.06	0.12	0.03	1.55	0.067
8	0.01	0.01	0.05	0.10	0.04	0.03	0.05	0.10	0.11	0.05	0.13	0.15	0.20	0.25	0.30	0.28	0.10	0.11	0.20	0.09	0.06	0.11	0.04	0.04	2.61	0.109
9	0.06	0.04	0.03	0.10	0.11	0.16	0.11	0.20	0.21		0.25	0.37	0.11	0.17	0.08	0.12	0.10	0.17	0.09	0.05	0.02	0.06	0.01	0.02	2.64	0.115
10	0.14	0.02	0.17	0.07	0.02	0.13	0.08	0.11	0.20	0.16	0.16	0.17	0.20	0.19	0.14	0.07	0.04	0.02	0.01	0.05	0.02	0.02	0.03	0.10	2.32	0.097
11	0.05	0.01	0.04	0.04	0.03	0.02	0.02	0.06	0.12	0.15	0.32	0.25		0.15	0.25			0.11		0.08					1.79	0.112
12																										
13										0.13	0.19	0.20	0.30	0.24	0.06	0.05	0.05	0.04	0.05	0.03	0.03	0.02	0.02	0.02	1.43	0.095
14	0.06	0.10	0.04	0.04	0.03	0.02	0.06	0.11	0.11	0.07	0.10	0.14	0.03	0.09	0.02	0.03	0.02	0.01	0.10	0.07	0.08	0.09	0.02	0.02	1.46	0.061
15	0.08	0.10	0.06	0.11	0.13	0.28	0.07						0.11	0.16	0.21	0.10	0.11	0.18	0.11	0.10	0.04	0.03	0.08	0.10	2.16	0.114
16	0.05	0.07	0.05	0.19	0.07	0.02	0.02	0.07	0.12	0.09	0.13	0.12	0.20	0.11	0.17	0.08	0.11	0.06	0.12	0.06	0.03	0.07	0.01	0.02	2.04	0.085
17	0.03	0.03	0.02	0.02	0.01	0.02	0.05	0.04	0.13	0.19	0.08	0.13	0.05	0.05	0.04	0.06	0.05	0.07	0.02	0.11	0.10	0.11	0.01	0.01	1.43	0.060
18	0.01	0.01	0.01	0.09	0.01	0.02	0.02	0.01	0.06	0.07	0.04	0.10	0.09	0.12	0.10	0.19	0.06	0.06	0.05	0.04	0.03	0.03	0.02	0.02	1.23	0.053
19	0.02	0.02	0.02	0.08						0.07	0.10	0.05	0.08	0.09	0.07	0.06	0.05	0.05	0.06	0.04	0.04	0.03	0.05	0.03	1.01	0.053
20	0.02	0.05	0.04	0.03	0.06	0.03	0.04	0.03	0.02	0.04	0.06	0.05	0.11	0.09	0.09	0.09	0.09	0.09	0.12	0.06	0.06	0.03	0.08	0.06	1.44	0.060
21	0.03	0.05	0.09	0.08	0.06	0.12	0.11	0.13	0.15	0.18	0.12	0.15	0.13	0.16	0.09	0.11	0.15	0.10	0.09	0.07	0.10	0.05	0.04	0.04	2.40	0.100
22	0.05	0.09	0.04	0.04	0.09	0.08	0.08	0.07	0.05	0.07	0.08	0.09	0.07	0.10	0.07	0.06	0.08	0.05	0.08	0.07	0.07	0.06	0.11	0.10	1.75	0.073
23	0.09	0.08	0.11	0.16	0.07	0.06	0.09	0.06	0.07	0.16	0.06	0.11	0.14	0.12	0.09	0.05	0.06	0.07	0.06	0.05	0.03	0.04	0.04	0.05	1.92	0.080
24	0.05	0.15	0.06	0.07	0.06	0.09	0.06	0.07	0.08																0.69	0.077
25																										
26															0.09	0.10	0.05	0.06	0.05	0.07	0.10	0.07	0.06	0.05	0.70	0.070
27	0.03	0.03	0.02	0.05	0.09	0.10	0.13	0.10	0.15	0.26	0.34	0.72	0.51	0.36	0.19	0.11	0.14	0.12	0.10	0.09	0.11	0.07	0.08	0.06	3.96	0.165
28	0.08	0.11	0.07	0.06	0.08	0.11	0.20	0.12	0.13	0.18	0.15	0.18	0.25	0.11	0.11	0.09	0.08	0.05	0.06	0.10	0.11	0.09	0.20	0.09	2.91	0.117
29	0.06	0.09	0.08	0.06	0.09	0.18	0.17	0.13	0.11	0.32	0.26	0.47	0.16	0.11	0.20	0.23	0.10	0.11	0.14	0.13	0.15	0.10	0.10	0.08	3.63	0.151
30	0.09	0.11	0.06	0.04	0.05	0.11	0.07	0.11	0.11	0.12	0.10	0.20	0.15	0.10	0.11	0.10	0.11	0.12	0.11	0.15	0.09	0.13	0.06	0.06	2.46	0.103
31	0.10	0.07	0.12	0.21	0.10	0.12	0.11	0.18	0.18																1.19	0.132
total	1.38	1.47	1.33	1.83	1.52	1.96	1.82	2.05	2.52	2.67	3.39	4.18	3.32	3.30	2.97	2.59	1.91	2.13	2.04	1.75	1.50	1.57	1.35	1.12	51.67	0.087
Mean	0.055	0.059	0.053	0.073	0.063	0.082	0.076	0.093	0.115	0.121	0.141	0.174	0.133	0.127	0.110	0.100	0.073	0.079	0.078	0.067	0.063	0.063	0.054	0.045	0.087	

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.05	0.03	0.03	0.03	0.04	0.04		0.02	0.08	0.10	0.08	0.07	0.02	0.02	0.03	0.03	0.04	0.06	0.03	0.03	0.03	0.02	0.03	0.04	0.95	0.041
2	0.03	0.03	0.23	0.18	0.08	0.07	0.02	0.08	0.10			0.10	0.12	0.08	0.12	0.10	0.07	0.09	0.09	0.03	0.04	0.07	0.08	0.02	1.83	0.083
3	0.02	0.02	0.01	0.02	0.05	0.07	0.06	0.10	0.07	0.15	0.06	0.08	0.09	0.02		0.05	0.06	0.04		0.03	0.02	0.03	0.02	0.03	1.10	0.050
4	0.07	0.09	0.02	0.02	0.04	0.04	0.07	0.05	0.02	0.05	0.09	0.03	0.06	0.03	0.02	0.01	0.03	0.02	0.05	0.03	0.04	0.10	0.08	0.07	1.13	0.047
5	0.02	0.02	0.02	0.03	0.02					0.05	0.03	0.02	0.05	0.03	0.03	0.02	0.06	0.06	0.05	0.09	0.03	0.10	0.02	0.02	0.77	0.039
6	0.03	0.10	0.02	0.02	0.02	0.04	0.04	0.05	0.05	0.03	0.07	0.02	0.03	0.02	0.03	0.02	0.03	0.03	0.02	0.05		0.02	0.05	0.03	0.82	0.036
7	0.04	0.07	0.03	0.05	0.06	0.02	0.04	0.02	0.03	0.08	0.08	0.07	0.07	0.02	0.02	0.08	0.08	0.10	0.03	0.10	0.03	0.10	0.05	0.02	1.29	0.054
8	0.02	0.02	0.03	0.05	0.02	0.04	0.10	0.08	0.10		0.03	0.07	0.03	0.01	0.04	0.02	0.09	0.04	0.02	0.01	0.01	0.02	0.03	0.02	1.90	0.083
9	0.03	0.05	0.02	0.02	0.06	0.05	0.01	0.03	0.02	0.05	0.02	0.03					0.04	0.06	0.03	0.03	0.05	0.02			0.62	0.034
10																										
11																										
12																										
13																										
14													0.04	0.05	0.03	0.05	0.02	0.02	0.06	0.05	0.05	0.06	0.04	0.04	0.51	0.043
15	0.05	0.04	0.04	0.03	0.04	0.03	0.06	0.04	0.06	0.02	0.03	0.02	0.02	0.05	0.02	0.02	0.02	0.02	0.03	0.03	0.03	0.02	0.03	0.02	0.77	0.032
16	0.02	0.01	0.08	0.14	0.04	0.10	0.07	0.07	0.15	0.07	0.03	0.06	0.03	0.02	0.03	0.02	0.03	0.03	0.03	0.03	0.02	0.04	0.03	0.05	1.19	0.050
17	0.03	0.04	0.06	0.09	0.05	0.06	0.02	0.23	0.10	0.12	0.05	0.08	0.09	0.04	0.02	0.04	0.05	0.07	0.08	0.05	0.02	0.05	0.02	0.01	1.47	0.061
18	0.03	0.02	0.02	0.03	0.05	0.02	0.03	0.05	0.06	0.10	0.07	0.08	0.02	0.03	0.02	0.03	0.08	0.05	0.03	0.03	0.03	0.10	0.05	0.04	1.07	0.045
19	0.02	0.06	0.05	0.03	0.04	0.05	0.07	0.03	0.07														0.03	0.02	0.47	0.043
20	0.03	0.05	0.01	0.01	0.02	0.02	0.03	0.02	0.05					0.04	0.03	0.02	0.02	0.02	0.03	0.06	0.14	0.08	0.03	0.02	0.73	0.037
21	0.03	0.02	0.03	0.02						0.03	0.07	0.02	0.04	0.02	0.02	0.03	0.03	0.04	0.02	0.03	0.01	0.02	0.01	0.01	0.50	0.026
22	0.02	0.03	0.04	0.08	0.02	0.02	0.01	0.05	0.02												0.03	0.03	0.03	0.02	0.40	0.031
23	0.02	0.02	0.01	0.02	0.03	0.02	0.04	0.02		0.11	0.02	0.05	0.09	0.08	0.10	0.04	0.08	0.09	0.10	0.11	0.10	0.05	0.04	0.07	1.31	0.057
24	0.10	0.03	0.10	0.04	0.07	0.03	0.05	0.05	0.03																0.50	0.056
25																										
26	0.03	0.01	0.02	0.05	0.04	0.03	0.03	0.02	0.02	0.02															0.27	0.027
27												0.09	0.03	0.02	0.03	0.10	0.02	0.02	0.08	0.05	0.12	0.10	0.02	0.04	0.72	0.055
28	0.03	0.01	0.05	0.02	0.02	0.05	0.02	0.02	0.05	0.06	0.02	0.05	0.07	0.08	0.05	0.02	0.01	0.01	0.02	0.01	0.07	0.03	0.04	0.02	0.83	0.038
29	0.01	0.01	0.04	0.02	0.02	0.02	0.01	0.02	0.02		0.06	0.02	0.05	0.02	0.01	0.04	0.02	0.01	0.03	0.02	0.01	0.02	0.01	0.02	0.51	0.022
30	0.01	0.06	0.01	0.01	0.02	0.01	0.02	0.01	0.05	0.05	0.06	0.04	0.02	0.02	0.01	0.02	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.03	0.58	0.024
31	0.02	0.01	0.01	0.05	0.05	0.06	0.02	0.03	0.01	0.02	0.02	0.03	0.02	0.02	0.02	0.02	0.02	0.03	0.01	0.03	0.03	0.02	0.05	0.03	0.63	0.026
total	0.79	0.84	0.98	1.06	0.90	0.89	0.82	1.09	1.16	1.11	0.89	1.02	0.99	0.72	0.68	0.78	0.91	0.92	0.85	0.92	0.93	1.12	0.81	0.69	21.87	0.044
Mean	0.033	0.035	0.041	0.044	0.039	0.040	0.039	0.050	0.055	0.063	0.049	0.051	0.050	0.034	0.034	0.037	0.041	0.042	0.040	0.042	0.042	0.049	0.035	0.030	0.042	

(N. S.) 1942, Sept.

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.02	0.02	0.02	0.01	0.01	0.02	0.01	0.02	0.05	0.04	0.04	0.05	0.02	0.03	0.02	0.01	0.02	0.06	0.02	0.08	0.28	0.07	0.07	0.09	1.08	0.045
2	0.05	0.06	0.03	0.10	0.02	0.09	0.02	0.01	0.02	0.04	0.04	0.03	0.02	0.03	0.02	0.03	0.07	0.04	0.02	0.01	0.02	0.07	0.03	0.03	1.90	0.079
3	0.04	0.03	0.03	0.02	0.02	0.03	0.02	0.03	0.10	0.08	0.02	0.03							0.02	0.03	0.12	0.01	0.01	0.03	0.67	0.037
4	0.02	0.02	0.05	0.02	0.08	0.02	0.05	0.05	0.02	0.02	0.02	0.03													0.40	0.033
5														0.02	0.01	0.02	0.05	0.03	0.02	0.03	0.01	0.03	0.02	0.01	0.25	0.023
6	0.05	0.02	0.02	0.01	0.03	0.04	0.02	0.02	0.01	0.02	0.04	0.05	0.02	0.03	0.02	0.03	0.08	0.02	0.02	0.03	0.03	0.06	0.04	0.02	0.73	0.024
7	0.01	0.04	0.03	0.02	0.10	0.06	0.01	0.02	0.04	0.08	0.07	0.02	0.04	0.02	0.03	0.02	0.01	0.03	0.01	0.02	0.07	0.06	0.01	0.02	0.84	0.035
8	0.03	0.01	0.02	0.01	0.01	0.01	0.01	0.03	0.01	0.03	0.07	0.05	0.05	0.01	0.03	0.02	0.01	0.02	0.01	0.02	0.01	0.02	0.03	0.01	0.61	0.025
9	0.02	0.01	0.03	0.02	0.05	0.02	0.03	0.06	0.01	0.02	0.02	0.01	0.02	0.04	0.01	0.01	0.02	0.05	0.08	0.07	0.04	0.02	0.03	0.01	0.70	0.029
10	0.01	0.02	0.02	0.10	0.01	0.04	0.06	0.01	0.02																0.29	0.032
11													0.05	0.04	0.03	0.03	0.02	0.05	0.06	0.01	0.02	0.02	0.02	0.01	0.36	0.030
12	0.01	0.01	0.08	0.01	0.01	0.01	0.02	0.05	0.03	0.01	0.06	0.02	0.26	0.23	0.20	0.18	0.15	0.18	0.14	0.08	0.23	0.06	0.09	0.03	2.15	0.090
13	0.04	0.06	0.02	0.04	0.02	0.06	0.10	0.07	0.07	0.20	0.10	0.04	0.07	0.15	0.12	0.09	0.05	0.10	0.07	0.10	0.05	0.06	0.03	0.07	1.80	0.075
14	0.07	0.02	0.15	0.08	0.15	0.04	0.03	0.10	0.19	0.25	0.23	0.14	0.30	0.20	0.06	0.10	0.09	0.18	0.09	0.08	0.15	0.04	0.09	0.09	2.92	0.121
15	0.15	0.05	0.02	0.03	0.03	0.03	0.15	0.06	0.16	0.19	0.08	0.20	0.12	0.10	0.15	0.19	0.08	0.12	0.09	0.08	0.16	0.02	0.09	0.12	2.37	0.097
16	0.04	0.02	0.04	0.20	0.06	0.05	0.07	0.08	0.02	0.04	0.09	0.08	0.05	0.09	0.06	0.03	0.05	0.07	0.05	0.12	0.08	0.07	0.06	0.09	1.61	0.067
17	0.04	0.04	0.01	0.05	0.07	0.10	0.20	0.15	0.30	0.27	0.20	0.14	0.20	0.05	0.25	0.20	0.12	0.12	0.12	0.10	0.14	0.28	0.13	0.02	3.30	0.138
18	0.25	0.08	0.10	0.09	0.10	0.15	0.20	0.20	0.18	0.30	0.26	0.27	0.20	0.16	0.08	0.10	0.10	0.18	0.08	0.10	0.20	0.12	0.10	0.10	3.70	0.154
19	0.09	0.12	0.03	0.20	0.05	0.12	0.17	0.12	0.15	0.25	0.15	0.13	0.18	0.09	0.12	0.10	0.12	0.08	0.12	0.18	0.10	0.04	0.13	0.05	2.89	0.120
20	0.05	0.03	0.08	0.06	0.03	0.06	0.10	0.08	0.09	0.10	0.03	0.09	0.02	0.10	0.09	0.06	0.07	0.10	0.08	0.07	0.12	0.14	0.04	0.07	1.76	0.073
21	0.15	0.12	0.08	0.05	0.07	0.03	0.06	0.10	0.15	0.08	0.11	0.14													1.14	0.095
22																										
23																										
24											0.03	0.04	0.03	0.13	0.07	0.07	0.03	0.10	0.01	0.02	0.02	0.02	0.01	0.01	0.59	0.042
25	0.09	0.09	0.01	0.01	0.01	0.02	0.02						0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.02	0.01	0.01		0.38	0.021
26																										
27																										
28													0.02	0.16	0.01	0.02	0.01	0.02	0.02	0.01	0.02	0.02	0.03	0.01	0.35	0.092
29	0.01	0.02	0.02	0.01	0.02	0.01	0.03	0.02	0.02	0.03	0.03	0.02	0.06	0.03	0.02	0.03	0.05	0.01	0.02	0.01	0.02	0.06	0.02	0.01	0.58	0.024
30	0.01	0.01	0.01	0.10	0.01	0.01	0.02	0.06	0.02	0.09	0.06														0.40	0.036
31																										
total	1.25	0.90	0.90	1.24	0.96	1.02	1.40	1.39	1.66	2.14	1.75	1.60	1.64	1.72	1.41	1.36	1.24	1.57	1.16	1.26	1.91	1.30	1.09	0.90	32.77	0.061
Mean	0.057	0.041	0.041	0.056	0.044	0.046	0.64	0.066	0.079	0.107	0.083	0.080	0.082	0.082	0.067	0.065	0.059	0.075	0.053	0.057	0.087	0.059	0.049	0.043	0.064	

Hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1																										
2																										
3																										
4														0.05	0.08	0.11	0.13	0.05	0.08	0.03	0.06	0.07	0.04	0.03	0.73	0.066
5	0.06	0.10	0.12	0.03	0.02	0.01	0.03	0.03	0.05	0.04	0.03	0.02	0.03	0.03		0.03	0.02	0.02	0.08	0.02	0.05	0.05	0.03	0.02	0.95	0.041
6	0.08	0.03	0.03	0.06	0.02	0.10	0.05	0.04	0.08	0.07	0.06	0.08	0.07	0.03	0.03	0.05	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.02	1.05	0.044
7	0.02	0.01	0.02	0.06	0.02	0.03	0.03	0.02	0.10	0.05	0.05			0.02	0.03	0.03	0.09	0.04	0.02	0.06	0.03	0.04	0.05	0.03	0.85	0.039
8	0.02	0.02	0.02	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.02	0.01	0.05	0.02	0.06	0.02	0.02	0.02	0.02	0.03	0.02	0.54	0.023
9	0.03	0.02	0.02	0.03	0.01	0.02	0.02	0.01	0.02	0.02	0.02	0.02	0.02	0.02		0.02	0.03	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.49	0.021
10	0.03	0.04	0.02	0.02	0.06	0.02	0.02	0.02	0.03	0.05	0.03		0.02	0.03	0.02	0.05	0.02	0.02	0.03	0.09	0.02	0.05	0.04	0.02	0.75	0.033
11	0.06	0.02	0.01	0.02	0.03	0.02	0.04	0.04	0.03	0.08	0.08	0.07	0.06	0.05	0.02	0.06	0.07	0.03	0.05	0.04	0.02	0.02	0.04	0.02	1.98	0.040
12	0.04	0.02	0.03	0.04	0.02	0.12	0.09	0.05	0.03	0.15	0.11	0.11	0.10	0.02	0.06	0.05	0.08	0.05	0.04	0.02	0.02	0.02	0.02	0.03	1.33	0.055
13	0.05	0.03	0.06	0.05	0.02	0.02	0.06	0.10	0.08	0.05	0.30	0.15	0.12	0.17	0.08	0.20	0.18	0.10	0.07	0.08	0.10	0.05	0.07	0.07	2.26	0.094
14	0.05	0.04	0.02	0.02	0.02	0.01	0.10	0.08	0.09	0.05	0.03	0.08	0.15	0.08	0.10		0.05	0.05	0.04	0.02	0.05	0.10	0.07	0.05	1.35	0.059
15	0.08	0.06	0.03	0.12	0.07	0.05	0.05	0.04	0.18	0.20	0.30	0.25	0.22	0.19	0.18	0.06	0.03	0.12	0.10	0.05	0.05	0.09	0.02	0.06	2.60	0.108
16	0.17	0.04	0.10	0.13	0.10	0.08	0.03	0.04	0.02	0.06	0.11	0.13	0.04	0.05	0.05	0.04	0.02	0.03							1.24	0.069
17																		0.08	0.07	0.04	0.06	0.03	0.07	0.06	0.41	0.059
18	0.02	0.06	0.03	0.03	0.05	0.06	0.03	0.08	0.08	0.10	0.08	0.08	0.13	0.07	0.07	0.03	0.01	0.10	0.08	0.05	0.04	0.02	0.03	0.03	1.56	0.059
19	0.03	0.05	0.01	0.10	0.05	0.06	0.02	0.04	0.08	0.18	0.12	0.07	0.08		0.05	0.06	0.07	0.03	0.03	0.10	0.08	0.10	0.01	0.10	1.52	0.066
20	0.12	0.02	0.15	0.06	0.01	0.01	0.08	0.15	0.02	0.04	0.09	0.06	0.10	0.08	0.10	0.03	0.08	0.13	0.05	0.07	0.05	0.15	0.03	0.03	1.61	0.067
21	0.05	0.02	0.02	0.06	0.03	0.07	0.05	0.05	0.03	0.04	0.13	0.06	0.04	0.05	0.04	0.01	0.02	0.01	0.02	0.02	0.02	0.03	0.06	0.05	0.98	0.041
22	0.03	0.03	0.03	0.03	0.01	0.09	0.04	0.06	0.10	0.08	0.08	0.10	0.05	0.04	0.06	0.06	0.03	0.07	0.05	0.08	0.07	0.08	0.06	0.03	1.38	0.058
23	0.01	0.06	0.06	0.09	0.03	0.03	0.02	0.05	0.02	0.03	0.07	0.10	0.05	0.03	0.04	0.03		0.02	0.05	0.01	0.03	0.05	0.02	0.01	1.90	0.083
24	0.07	0.02	0.06	0.02	0.01	0.03	0.04	0.06	0.05	0.03	0.06	0.13	0.04	0.03	0.05	0.02	0.07	0.07	0.06	0.09	0.09	0.06	0.09	0.07	1.30	0.054
25	0.03	0.05	0.05	0.05	0.02	0.02	0.02	0.02	0.02	0.02	0.02		0.03	0.04	0.07	0.03	0.03	0.03	0.02	0.01	0.03	0.04	0.01	0.01	0.67	0.029
26	0.03	0.02	0.02	0.06	0.04	0.07	0.06	0.08	0.07	0.10	0.10	0.07	0.06	0.04	0.05	0.03	0.03	0.02	0.02	0.01	0.01	0.01	0.02	0.04	1.06	0.044
27	0.02	0.03	0.06	0.07	0.02	0.02	0.05	0.06	0.03	0.07	0.10	0.07	0.04	0.07	0.03	0.07	0.01	0.02	0.03	0.02	0.02	0.02	0.02	0.05	0.98	0.041
28	0.06	0.03	0.07	0.03	0.02	0.03	0.03	0.05	0.04	0.07	0.10	0.08	0.03	0.08	0.03	0.03	0.08	0.05	0.05	0.07	0.08	0.04	0.07	0.13	1.35	0.056
29	0.19	0.10	0.20	0.10	0.11	0.07	0.12	0.05	0.08	0.10	0.17	0.10	0.20	0.15	0.10	0.13	0.20	0.30	0.07	0.10	0.10	0.16	0.10	0.03	3.03	0.126
30	0.12	0.07	0.09	0.07	0.05	0.10	0.05	0.19	0.10	0.07	0.20	0.10	0.12	0.12	0.02	0.06	0.20	0.08	0.23	0.10	0.10	0.06	0.10	0.07	2.87	0.120
31	0.03	0.13	0.05	0.10	0.10	0.08	0.12	0.08	0.20	0.16	0.30	0.11	0.15	0.20	0.13	0.05	0.07	0.06	0.06	0.05	0.18	0.11	0.05	0.22	2.79	0.116
Total	1.50	1.15	1.36	1.64	1.15	1.27	1.27	1.41	1.64	1.93	2.76	2.08	1.97	1.76	1.50	1.39	1.66	1.68	1.46	1.29	1.42	1.51	1.21	1.32	37.33	0.061
Mean	0.048	0.041	0.052	0.063	0.044	0.049	0.049	0.054	0.063	0.074	0.106	0.080	0.079	0.068	0.060	0.053	0.064	0.060	0.054	0.048	0.053	0.056	0.045	0.049	0.060	

(N. S.) 1942, Nov.

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.03	0.05	0.04	0.02	0.02	0.02	0.03	0.05	0.07	0.08	0.03	0.10	0.10	0.03	0.02	0.03	0.02	0.02	0.03	0.07	0.02	0.02	0.05	0.02	0.97	0.040
2	0.02	0.18	0.05	0.03	0.06	0.03	0.02	0.03	0.10	0.06	0.04	0.05	0.03		0.02	0.08	0.03	0.04	0.01	0.06	0.02	0.02	0.02	0.05	1.05	0.046
3	0.19	0.02	0.02	0.04	0.03	0.02	0.05	0.06	0.07	0.03	0.02	0.02	0.03	0.06	0.04	0.06	0.03	0.03	0.03	0.08	0.10	0.06	0.02	0.02	1.13	0.047
4	0.03	0.03	0.04	0.02	0.02	0.05	0.02	0.04	0.03				0.02	0.03	0.04	0.03	0.01	0.06	0.03	0.02	0.04	0.03	0.02	0.02	0.63	0.030
5	0.03	0.02	0.03	0.04	0.02	0.01	0.02	0.03	0.05	0.04	0.02	0.02	0.05	0.09	0.04	0.05	0.02	0.02	0.03	0.02	0.04	0.02	0.01	0.02	0.74	0.031
6	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.06	0.09	0.08	0.05	0.02	0.07	0.02	0.04	0.02	0.02	0.01	0.02	0.03	0.02	0.03	0.02	0.05	0.80	0.033
7	0.08	0.05	0.03	0.02	0.03	0.03	0.03	0.03	0.02	0.08	0.05	0.07	0.06	0.07	0.03	0.02	0.03	0.05	0.07	0.03	0.06	0.04	0.03	0.03	1.04	0.043
8	0.02	0.02	0.03	0.03	0.04	0.05	0.02	0.04	0.02	0.08	0.10	0.02	0.08	0.05	0.06		0.02	0.03	0.05	0.08	0.05	0.04	0.03	0.03	0.99	0.043
9	0.03	0.05	0.04	0.05	0.03	0.03	0.05	0.02	0.03	0.12	0.35	0.40	0.15	0.06	0.02	0.10	0.02	0.03	0.02	0.03	0.03	0.02	0.01	0.03	1.72	0.072
10	0.03	0.02	0.02	0.02	0.01	0.02	0.03	0.02	0.04	0.08	0.12	0.03	0.02	0.08	0.05	0.03	0.03	0.02	0.07	0.02	0.04	0.05	0.02	0.05	0.92	0.038
11	0.07	0.04	0.10	0.09	0.02	0.06	0.09	0.16	0.23	0.28	0.22	0.10	0.01	0.07	0.04	0.02	0.06	0.07	0.07	0.03	0.05	0.02	0.02	0.02	1.94	0.081
12	0.01	0.02	0.06	0.10	0.06	0.05	0.05	0.10	0.40	0.10	0.08		0.05	0.10	0.06	0.05	0.10	0.03	0.02	0.01	0.03	0.01	0.01	0.01	1.51	0.061
13	0.04	0.01	0.02	0.03	0.06	0.02	0.03	0.02	0.03	0.10	0.13	0.06	0.02	0.10	0.05	0.03	0.05	0.02	0.02	0.01	0.07	0.06	0.03	0.02	1.03	0.043
14	0.03	0.05	0.03	0.02	0.03	0.02	0.03	0.05	0.03	0.10	0.07	0.20	0.02	0.05	0.05	0.12	0.08	0.03	0.03	0.02	0.08	0.03	0.08	0.05	1.30	0.054
15	0.02	0.02	0.04	0.09	0.06	0.09	0.03	0.06	0.03	0.10	0.13	0.18	0.12	0.13	0.09	0.06	0.02	0.02	0.02	0.02	0.02	0.03	0.02	0.03	1.43	0.060
16	0.03	0.02	0.07	0.03	0.09	0.11	0.11	0.07	0.10	0.10	0.03	0.09	0.02	0.03	0.03	0.10	0.02	0.02	0.01	0.02	0.03	0.03	0.03	0.02	1.21	0.050
17	0.02	0.02	0.02	0.02	0.03	0.02	0.06	0.02	0.03	0.04	0.04	0.02	0.02	0.09	0.05	0.05	0.02	0.02	0.03	0.02	0.02	0.06	0.01	0.02	0.75	0.031
18	0.06	0.03	0.05	0.03	0.02	0.02	0.04	0.02	0.04	0.03	0.03	0.10	0.06	0.04	0.08	0.07	0.05	0.02	0.05	0.01	0.03	0.03	0.03	0.02	0.96	0.040
19	0.08	0.07	0.02	0.06	0.05	0.05	0.06	0.06	0.03	0.15	0.08	0.09	0.10	0.07	0.09	0.06	0.10	0.05	0.05	0.06	0.05	0.09	0.02	0.06	1.60	0.067
20	0.07	0.04	0.06	0.02	0.03	0.02	0.02	0.05	0.03	0.03	0.04	0.07	0.03	0.03	0.05	0.06	0.10	0.06	0.05	0.07	0.04	0.03	0.02	0.02	1.04	0.043
21	0.04	0.03	0.03	0.22	0.07	0.02	0.02	0.02	0.03	0.21	0.05	0.03	0.09	0.04	0.09	0.09	0.06	0.02	0.02	0.08	0.02	0.02	0.03	0.02	1.35	0.056
22	0.05	0.06	0.07	0.06	0.03	0.04	0.07	0.05	0.03	0.03	0.09	0.07	0.08	0.06	0.03	0.04	0.02	0.05	0.02	0.05	0.06	0.15	0.02	0.03	1.26	0.053
23	0.03	0.02	0.05	0.02	0.03	0.02	0.02	0.03	0.02	0.05	0.03	0.05	0.03	0.04	0.03	0.03	0.04	0.05	0.04	0.04	0.05	0.06	0.07	0.05	0.90	0.038
24	0.10	0.02	0.03	0.03	0.02	0.04	0.12	0.15	0.20	0.10	0.10	0.05	0.08	0.10	0.10	0.19	0.09	0.03	0.10	0.08	0.06	0.10	0.08	0.10	2.07	0.086
25	0.11	0.02	0.11	0.04	0.03	0.02	0.08	0.05	0.10	0.07	0.11	0.11	0.03	0.07	0.05	0.03	0.07	0.10	0.10	0.06	0.11	0.05	0.07	0.10	1.69	0.070
26	0.09	0.03	0.05	0.04	0.06	0.05	0.12	0.05	0.09	0.26	0.12	0.07	0.12	0.06	0.10	0.08	0.05	0.05	0.02	0.10	0.15	0.12	0.15	0.05	2.08	0.087
27	0.03	0.07	0.10	0.08	0.04	0.06	0.09	0.03	0.02	0.10	0.08	0.05	0.02	0.04	0.03	0.05	0.07	0.03	0.04	0.02	0.02	0.03	0.04	0.03	1.17	0.049
28	0.03	0.02	0.02	0.02	0.05	0.05	0.03	0.02	0.10	0.04	0.03	0.02					0.02	0.04	0.02	0.05	0.02	0.03	0.05	0.02	0.68	0.034
29	0.03	0.04	0.01	0.02	0.05	0.04	0.02	0.02	0.05	0.04	0.02	0.02	0.02	0.05	0.07	0.05	0.02	0.06	0.05	0.07	0.03	0.01	0.01	0.01	0.81	0.034
30	0.03	0.07	0.05	0.02	0.03	0.05	0.06	0.06	0.05	0.04	0.02				0.02	0.03	0.02	0.02	0.01	0.02	0.02	0.05	0.02	0.04	0.73	0.035
31																										
total	1.45	1.16	1.31	1.33	1.14	1.13	1.45	1.47	2.16	2.62	2.28	2.11	1.53	1.66	1.47	1.63	1.29	1.10	1.13	1.28	1.38	1.34	1.04	1.04	35.50	0.050
Mean	0.048	0.039	0.044	0.044	0.038	0.038	0.048	0.049	0.072	0.090	0.079	0.078	0.055	0.061	0.051	0.058	0.043	0.037	0.038	0.043	0.046	0.045	0.035	0.035	0.051	

hour / day	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	total	Mean
1	0.01	0.03	0.04	0.02	0.02	0.01	0.01	0.01	0.04	0.02	0.02	0.04	0.02		0.03	0.01	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.05	0.52	0.023
2	0.03	0.03	0.02	0.02	0.01	0.02	0.01	0.01	0.02	0.02	0.02	0.03	0.02	0.06	0.04	0.02	0.01	0.01	0.02	0.02	0.03	0.02	0.01	0.05	0.55	0.023
3	0.03	0.02	0.03	0.02	0.02	0.02	0.05	0.02	0.03	0.03	0.04	0.03	0.02	0.01			0.01	0.02	0.02	0.03	0.01	0.05			0.51	0.026
4			0.03	0.01	0.02	0.04	0.01	0.02	0.04			0.04	0.03	0.02	0.05	0.02	0.02	0.04	0.06	0.01	0.01	0.02	0.03	0.01	0.53	0.027
5	0.03	0.02	0.02	0.03	0.01	0.01	0.02	0.02	0.05	0.04	0.10	0.05	0.03	0.02	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.02	0.01	0.01	0.65	0.027
6	0.01	0.02	0.03	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.02	0.02	0.02	0.03	0.02	0.04	0.54	0.023
7	0.02	0.01	0.02	0.02	0.03	0.02	0.05	0.02	0.03	0.02	0.03	0.03	0.02	0.02	0.04	0.02	0.03	0.02	0.02	0.02	0.03	0.02	0.03	0.03	0.60	0.025
8	0.03	0.10	0.05	0.04	0.02	0.02	0.08	0.02	0.05	0.04	0.05	0.02	0.03	0.02	0.02	0.02	0.02	0.01	0.03	0.02	0.03	0.02	0.03	0.02	0.79	0.033
9	0.03	0.04	0.03	0.03	0.03	0.04	0.03	0.06	0.03	0.09	0.10	0.05													0.56	0.047
10													0.09	0.08	0.11	0.03	0.04	0.05	0.05	0.05	0.02	0.02	0.01	0.02	0.57	0.048
11	0.03	0.02	0.03	0.02	0.02	0.04	0.06	0.04	0.03	0.05	0.05	0.03	0.05	0.04	0.08	0.12	0.03	0.05	0.02	0.07	0.06	0.10	0.03	0.10	1.17	0.049
12	0.03	0.02	0.05	0.06	0.07	0.02	0.05	0.07	0.08	0.06	0.10	0.07	0.03	0.06	0.07	0.06	0.06	0.04	0.09						1.09	0.057
13																0.03	0.04	0.03	0.03	0.03	0.04	0.02	0.06	0.03	0.31	0.034
14	0.04	0.02	0.07	0.02	0.09	0.04	0.02								0.10	0.03	0.04	0.08	0.02	0.03	0.06	0.03	0.02	0.05	0.76	0.045
15	0.04	0.04	0.02	0.01	0.02	0.03	0.02	0.10	0.01	0.09	0.03	0.03	0.04	0.04											0.52	0.037
16																										
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23																										
24																										
25																										
26																	0.03	0.03	0.03	0.03	0.05	0.06	0.10	0.12	0.45	0.056
27	0.05	0.01	0.03	0.03	0.05	0.04	0.05	0.03	0.08	0.05	0.04	0.03		0.02	0.01	0.02	0.04	0.03	0.02	0.03	0.05	0.03	0.04	0.05	0.89	0.039
28	0.05	0.03	0.04	0.03	0.03	0.02	0.02	0.03	0.02	0.02	0.03	0.05	0.06	0.04	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.02	0.03	0.03	0.62	0.026
29	0.03	0.02	0.03	0.02	0.01	0.01	0.01	0.02	0.02	0.03	0.02	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.38	0.016
30	0.04	0.03	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.02	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.02	0.43	0.018
31	0.02	0.01	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02										0.18	0.012
total	0.53	0.39	0.58	0.49	0.50	0.43	0.53	0.51	0.57	0.61	0.69	0.56	0.49	0.49	0.66	0.46	0.46	0.52	0.52	0.43	0.47	0.51	0.47	0.64	12.62	0.033
Mean	0.051	0.029	0.052	0.027	0.028	0.024	0.029	0.030	0.034	0.038	0.043	0.033	0.031	0.31	0.044	0.029	0.026	0.029	0.029	0.025	0.028	0.030	0.029	0.040	0.031	

MAGNETIC DIP IN KUMAMOTO PREFECTURE

Saemon Tarô NAKAMURA, & Akira ONUKI

(Received Jan. 10, 1955)

Systematic investigation of the earth's magnetism within Kumamoto Prefecture was executed by the authors since 1952, being aided by scientific research expenditure of the Minister of Education. The Results obtained in this survey were listed in the following table, in which calculated values of anomalies for each stations from the standard general magnetic field are also given. The standard general field was determined by the method of least squares.

The dip circle was used for the determination. At each station the measurements were repeated two or three times at least, and it was assured that the each measurement coincides within an error not exceeding 3′.

For those stations which were considered as important for the investigation, for example such as Kumamoto and Yatusiro, measurements were repeated during these 3-years, which show small secular changes, which do not exceed much more than 10′. It suggests that the observed values enlisted in the table are, being the date of measurements not the same, not entirely free from the secular fluctuation. But, being the purpose of this paper to determine the general tendency of the anomalies all over the district, in the above table only the mean of all values obtained for that place during the whole period are enlisted. Denoting by l, φ, θ, the values of longitude−130°, latitude−32°, and Dip−45° respectedly, we have assumed the quadratic equation for the standard magnetic dip and obtained.

$$\theta = 13.0664 + 2.0733\varphi - 0.0106\varphi^2 - 0.0415l - 0.0025\varphi l$$

where l, φ, θ, are all expressed in minutes. The coefficients of φ^2 and φl are comparatively small, and it would be safely neglected in the rough approximation. All 100 stations except one were used for the reduction of the equation. The excluded one is No. 87 "Kinpô san", which is the summit of old volcano and shows exceedingly large anomaly, undoubtedly affected greatly by the volcanic minerals.

The anomalies, defined by the formula

$$v = \theta_{obs} - \theta_{cal}$$

are enlisted in the sixth column in the table, which show remarkable rapid changes in the middle eastern part of this prefecture, and of course, it seems that this is coming from the volcanic rock of Mt. Aso. Nevertheless, it is also conspicuous that the distribution of v has some correlation with the famous fault line connecting Ôita and Yatusiro. One of the authors has already proposed a theory to explain the distribution of v along the fault, near Yatusiro, assuming an under ground distribution of magnetic dipoles and has obtained the good result

(Ref.) Our investigation will throw some light upon the problem of prospecting of earthquake, geological or geophysical structure of earth's crust in this prefecture.

We express our heartful thanks to all persons, who gave us many important helps, especially for Shin Nippon Chisso Company, who kindly allowed to use lorry everywhere we wanted, and for Yuen-Tetsudô Company who kindly offered us free license for their all traffic lines. We are much indebted to Mr. M. Itô for his assistance in the observations.

MAGNETIC DIP IN KUMAMOTO PREFECTURE

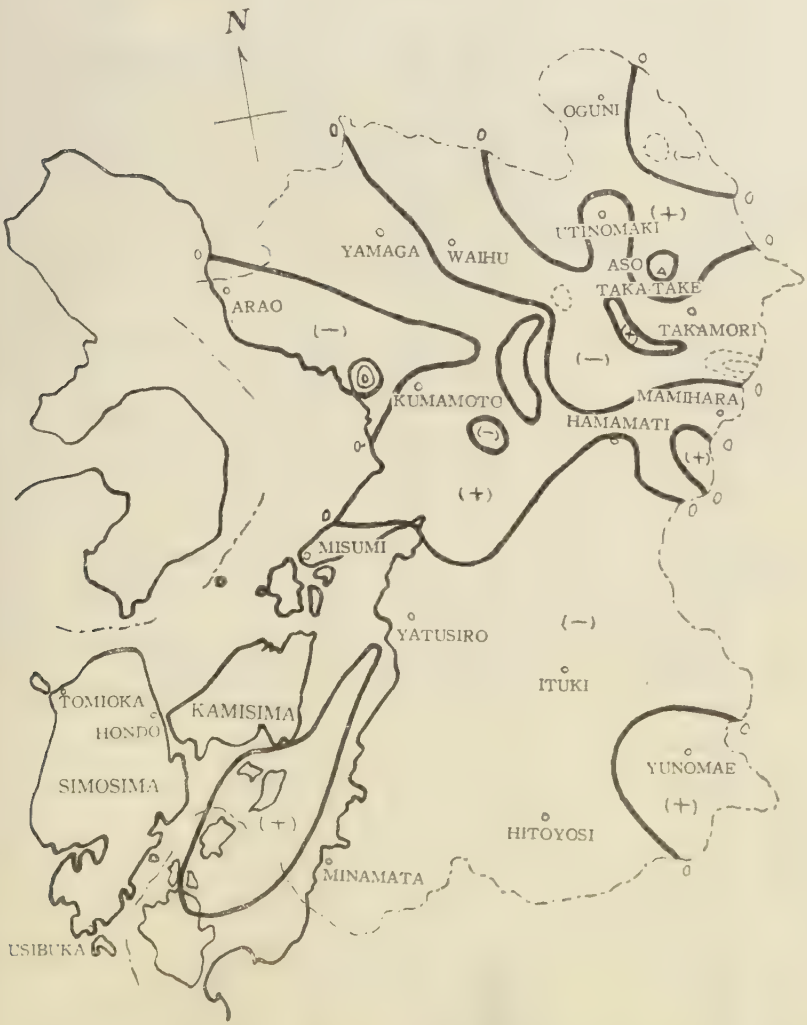


Table of Magnetic Dip.

No.	Station	ϕ	λ	θ	v	Remark
1	Usibuka	11.2	1.7	32.9	−1.9	
2	Wasi-no-Su	11.4	10.7	40.2	+5.6	Nagasima
3	Kukino	10.2	18.7	20.1	−11.8	Asikita
4	Kamenoura	16.2	2.7	42.4	−1.2	Amakusa Simozima
5	Hitoyosi	12.5	45.8	27.4	−6.6	
6	Takahama	21.7	−0.4	49.3	−3.8	
7	Sasiki	17.9	30.1	42.3	−1.9	
8	Yunomae	16.2	5?.2	41.4	+2.4	
9	Rankuti	21.0	21.4	66.5	+16.6	Gosyonoura
10	Tôzi	23.6	49.7	42.7	−8.4	
11	Hondo	27.5	10.9	56.8	−4.1	
12	Tomioka	31.6	1.3	63.3	−4.5	
13	Aizu	30.9	25.3	61.0	−3.0	
14	Yatusiro	30.5	36.0	61.4	−0.8	
15	Kureko	27.5	54.5	46.7	−9.4	
16	Siibaru	29.3	54.3	50.3	−8.2	
17	Momiki	30.3	57.8	40.2	−19.2	Gokanosyô
18	Kosigoe	32.7	56.5	58.9	−3.7	
19	Nihonsugi	34.4	54.1	63.5	−1.4	
20	Hayakusu	35.8	53.5	41.5	−25.2	
21	Tomoti (N)	37.1	52.0	99.4	+31.0	
22	Motoyasiki	35.7	69.1	56.3	−8.2	Miyazaki-ken
23	Kuraoka	38.4	69.4	85.9	+18.4	
24	Sirahuzi	39.1	59.4	86.1	+16.4	
25	Hamamati	40.8	59.6	80.6	+9.1	
26	Mamihara 1.	40.6	69.3	122.3	+52.4	Ryôgokubasi
27	Mamihara 2.	40.6	69.3	126.4	+56.5	
28	Mihune	42.7	48.1	59.2	−15.9	
29	Seto	43.8	53.7	110.7	+35.3	Nanataki
30	Nisemoto	44.3	70.9	−11.2	−84.5	
31	Simonagasaki	42.6	70.3	168.7	+96.9	
32	Gosyo	43.7	62.9	42.7	−31.2	Kamimasiki
33	Takamori	48.6	67.5	31.0	−46.8	
34	Kumamoto	48.3	43.2	84.4	+2.9	Kurokamityô
35	Tateno	52.3	58.3	38.9	+43.6	
36	Aso sanzyô	52.5	64.6	131.2	+49.7	
37	Miyazi	56.4	67.0	94.6	+10.6	Itinomiya
38	Utinomaki	58.4	62.5	71.6	−14.7	
39	Tuetate	70.6	62.2	113.1	+20.1	
40	Matuai	37.8	36.0	68.1	−3.3	
41	Matubase	39.9	40.2	75.3	+2.1	
42	Ôzu	52.2	52.4	110.5	+27.1	
43	Yamaga	60.9	41.8	119.9	+28.0	
44	Waihu	58.5	48.6	83.7	−5.2	
45	Simazaki	48.7	40.6	78.1	−4.2	
46	Kôsa	39.1	48.7	83.1	+12.0	
47	Wasyôzi	34.1	47.2	71.2	+5.7	
48	Akune	1.0	12.0	47.5	+32.9	
49	Imutabaru	43.6	66.2	68.5	−4.9	Kamimasiki
50	Yasukata	42.2	67.4	86.9	+15.1	
51	Yoneo	41.2	65.8	107.2	+36.2	
52	Kariya	40.8	66.4	61.0	−9.5	
53	Naidaizin	35.9	60.2	72.9	+7.0	
54	Turu	38.1	58.9	44.3	−24.3	Higasi Tomoti.
55	Setoisi	22.0	38.7	41.9	−7.9	
56	Kobusiki	40.1	55.3	85.6	+14.3	
57	Namazu	44.0	44.3	80.7	+3.7	
58	Kumanosyô	42.1	44.0	86.7	+11.6	
59	Aso Sirakawa	49.4	64.8	39.3	−39.6	
60	Kiyomizu	48.1	65.2	33.2	−44.5	

61	Kiyomizu Tôge	47.2	65.1	85.7	+8.8	
62	Seta	52.4	54.7	4.7	−78.5	
63	Nisikino	51.7	54.9	46.2	−36.3	
64	Mori	51.6	53.7	108.1	+25.4	
65	Kagami	33.6	39.6	58.6	−7.2	
66	Sanrigi	51.3	47.8	72.7	−10.7	
67	Nanamagari	51.0	58.9	104.7	+23.4	Aso Gairin
68	Takamori Tôge	48.0	68.5	36.8	−40.3	
69	Tabikusa	46.6	69.3	74.5	−1.2	
70	Kiyama	47.1	49.2	116.2	+36.8	
71	Tomoti (S)	37.1	52.0	59.3	−9.1	
72	Kawahara (1)	48.6	53.3	125.2	+45.1	Syôgakkô
73	Kawahara (2)	48.6	53.3	19.2	−60.9	Tyugakkô
74	Taki	48.5	53.7	189.3	+109.4	
75	Monde	48.5	53.3	50.8	−29.2	
76	Akita	48.5	53.0	88.5	+8.4	
77	Zyûmozi Tôge	47.2	56.9	66.7	−11.5	
78	Uto	40.6	36.2	76.2	+1.6	
79	Kawaziri	43.8	37.5	83.4	+5.5	
80	Nankan	63.4	31.7	96.9	+1.4	
81	Miibaru	63.4	46.2	90.0	−2.6	Utidamura
82	Osima	46.4	38.5	77.1	−3.3	
83	Oama	51.7	35.6	74.6	−11.2	
84	Tamana	55.6	33.8	78.5	−10.9	
85	Ueki	53.6	41.8	86.3	−0.1	
86	Takae	55.1	45.7	96.0	+9.1	
87	Kimpôsan	48.6	38.3	312.3	+229.8	
88	Hirakawa	54.2	53.7	58.8	−26.0	
89	Ihagi	56.7	51.3	84.3	−2.8	
90	Daikanbô	60.3	64.0	128.0	+40.8	
91	Miyabaru	66.9	64.3	99.0	+8.1	Ogunimati
92	Kurokawa	64.4	68.6	35.0	−53.7	
93	Kawaguti	42.9	37.6	80.6	+3.7	
94	Hahuzi	45.6	39.8	85.5	+6.1	
95	Yunoko	14.0	25.1	46.1	+8.0	
96	Minamata (1)	12.6	24.4	13.3	−22.4	Daiiti Syôgakkô
97	Nakamura	26.9	45.7	40.5	−15.7	Ituki
98	Yuyama	19.0	62.8	33.2	−9.8	
99	Huruyasiki	22.5	61.3	42.6	−5.8	Mizukamimura
100	Minamata (2)	12.3	25.0	28.5	−8.7	Zinnai.

ϕ = Latitude − 32° l = Longitude − 130° θ = Dip − 45°

$v = \theta - [13.0664 + 2.0733\phi - 0.01062\phi^2 - 0.04151l - 0.00249\phi l]$

ON THE SURFACE FIGURES OF THE EARTH, THE MOON AND MARS. (REPORT IV)

Saemon Taro NAKAMURA.

Part II. Zonal terms of the Surface Figures of the Earth, the Moon and Mars.

Chapter I. Data for the Surface Figures of the Moon and Mars. The surface elevation of the Moon was given by Franz, which was introduced in "Zusetu Tenmongaku Kôza" (Graphical Astronomy) by K. Murakami of Hiroshima University. The present author wrote to him wishing to see the original paper. But Murakami lost it by fire in 1945 at Hiroshima. The data here used were, therefore, read off from the Murakami's reproduction.

As in the case of the Earth the height is classified into 1, 0 and −1 according as elevated, mean and depressed. The hemispherical part facing to the earth only is available in this case. Accordingly, the result obtained must be less accurate than that obtained in the case of the earth. The data read off and used in the calculation are given in Table 6.

The surface elevation of Mars is unknown. The surface figure only known is the bright and dark parts, which are represented by 1 and -1 respectively. The data used are read off from E. M. Antoniadi's chart partly improved by S. Ebisawa using recent observations. Table 7 gives the data. The polar regions are excluded.

Chapter II. The Coefficients of Zonal Terms.

Using the data given in Tables 6 and 7, the coefficients of zonal terms are calculated. Table 8 gives the results. The number of free oscillations per second calculated by equation (5) is also given in the Table. For comparison, the data for the Earth obtained in Part I are reproduced in it.

As it was expected at the end of the last part, there is a maximum coefficient for Mars nearly at the same frequency as in the case of the Earth. But in the case of the Moon, there is a minimum value. Though this result can be said that it occurs merely by chance, but the following explanation may also be possible, if there is anything physical meaning of it.

There was some agency, probably from extra-solar system, causes oscillations with the same frequency of about 23×10^{-4} sec^{-1} on each of these bodies. The remains are found on the Earth and Mars at present. But the oscillation of the Moon has been quickly damped out by tidal action between the Earth and the energy has been absorbed by the Earth. Thus on the Earth, corresponding coefficient is extraordinary large.

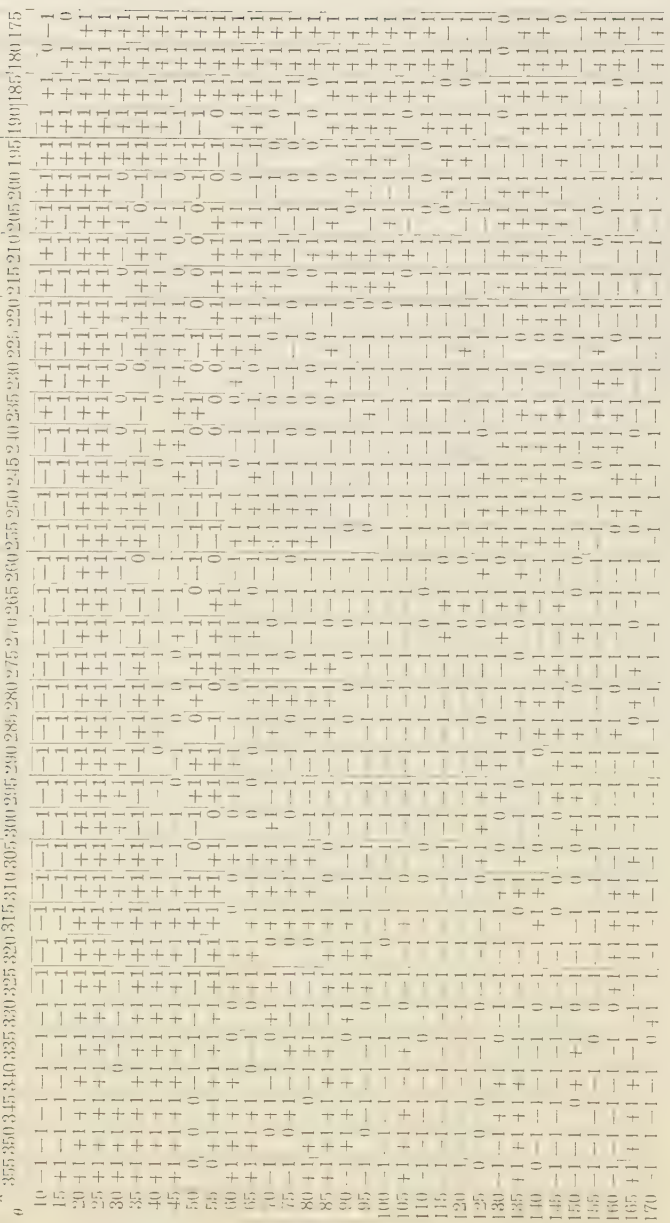
Table 6. Moon

λ Longitude
θ North-Polar Distance

θ \ λ	0	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	mean
0	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1.000
5	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0.540
10	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0.973
15	0	0	0	0	0	+1	+1	+1	+1	0	0	0	0	0	0	0	0	0	0	0.514
20	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.027
25	0	0	−1	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.297
30	−1	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.297
35	−1	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.351
40	0	0	0	+1	+1	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.189
45	0	+1	+1	+1	+1	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.351
50	−1	0	+1	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.622
55	−1	0	+1	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.730
60	1	0	+1	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.720
65	−1	0	+1	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.865
70	−1	0	+1	0	1	1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.811
75	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−0.568
80	+1	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	+1	+1	−1	−1	0	−0.189
85	+1	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	+1	+1	+1	+1	0	−0.081
90	+1	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	+1	+1	+1	+1	0	−0.054
95	+1	0	−1	−1	−1	−1	−1	−1	−1	0	0	0	+1	+1	+1	−1	0	+1	+1	0.162
100	0	0	−1	−1	−1	−1	−1	−1	−1	−1	0	+1	+1	+1	+1	+1	+1	+1	+1	0.405
105	1	0	−1	−1	−1	−1	−1	−1	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	0.432
110	1	0	−1	−1	−1	0	0	0	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	0.595
115	1	0	0	0	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	0.784
120	+1	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	0.919
125	+1	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	−1	0.891
130	+1	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	−1	−1	0.865
135	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	0	0	−1	0.730
140	1	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	−1	−1	−1	0	0.568
145	−1	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	0	0	+1	0.432
150	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	−1	−1	+1	0.405
155	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	0	0	−1	0	0.432
160	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0	0	0	0	0	0	0.703
165	0	0	0	0	0	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	0.811
170	1	0	1	1	−1	−1	0	0	0	0	+1	+1	+1	+1	+1	+1	+1	+1	+1	0.514
175	1	1	1	1	1	1	1	−1	−1	0	0	0	0	0	0	+1	+1	+1	+1	0.351
180	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	+1	1.000

Table 7. Mars

λ Longitude
θ North Polar Distance



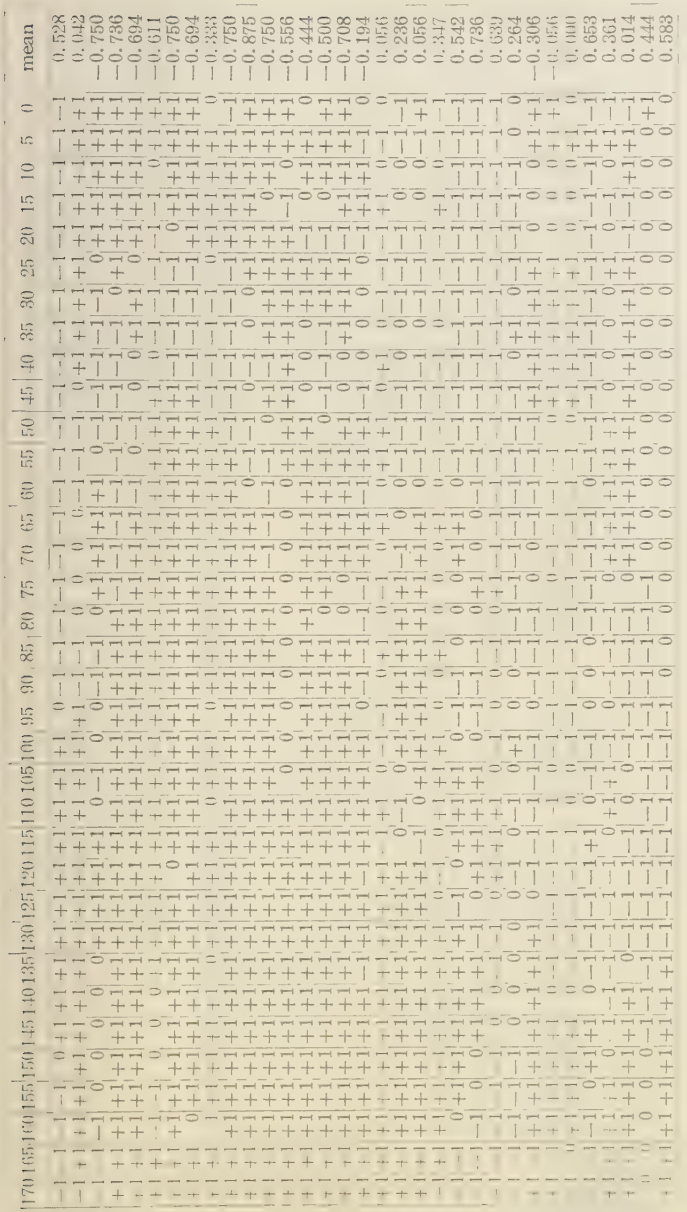

0 5 10 15 20 25 30 35 40 45 50 55 60 65 70 75 80 85 90 95 100 105 110 115 120 125 130 135 140 145 150 155 160 165 170
mean
0.528
0.042
−0.750
−0.736
0.694
0.611
0.750
0.694
0.333
0.750
0.875
0.750
0.556
0.444
0.500
0.708
0.194
0.056
0.236
0.056
0.347
0.542
0.736
0.639
0.264
0.306
0.056
0.000
0.653
0.361
0.014
0.444
0.583

Table 8

	Moon			Mars			Earth		
n	σ_n^2	σ_n	A_{on}	σ_n^2	σ_n	A_{on}	σ_n^2	σ_n	A_{on}
0	—	—	0.0826	—	—	0.1798	—	—	−0.4214
1	0×10^{-8}	0.0×10^{-4}	−0.7649	0×10^{-8}	0.0×10^{-4}	0.6507	0×10^{-8}	0.0×10^{-4}	0.9322
2	75	8.6	0.2416	88	9.4	0.0116	123	11.1	0.6568
3	160	12.6	0.8720	188	13.7	−0.5510	264	16.2	−0.3854
4	249	15.8	0.1332	293	17.1	0.1798	411	20.3	0.5737
5	339	18.4	−0.1748	400	20.0	0.2186	560	23.7	−1.2965
6	430	20.7	0.1291	507	22.5	0.5106	710	26.7	0.4485
7	522	22.8	0.0461	615	24.8	−0.2626	862	29.4	−0.7531
8	614	24.8	0.3028	724	26.9	0.1090			
9	706	26.6	0.3242	833	28.9	−0.5300			

THE TIDAL WAVES CAUSED BY TYPHOON ON THE WEST COAST OF KYUSHU (PART 1) (MISUMI HARBOR)

Yasuhiro SUGIMOTO

(Received Decenber 25, 1954)

Abstract

It may be supposed that the meteorological tide in the case of typhoon chiefly depends on the wind velocity and the barometric depression. The writer analyzed the mareograms at Misumi harbor and got the following expression:

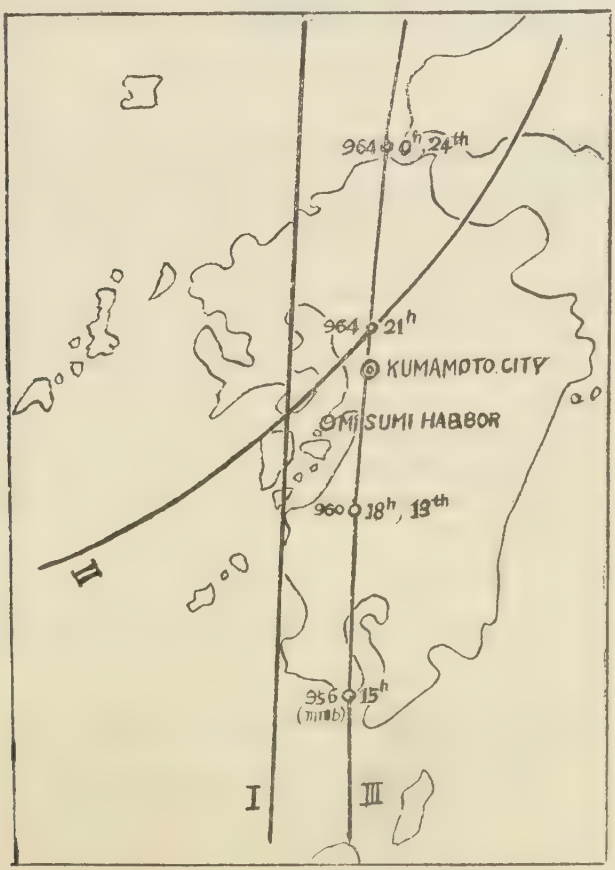


Fig. 1. The course of Typhoons. I: Jul. 25-26th 1937.
II: sep. 8-9t 1934. III: Sep. 13-14th 1954.

$$\Delta H = 1.39\Delta P + 3.15W_s + 0.68W_w$$

where ΔH: height of meteorological tide in cm.

ΔP : barometeric depression in mm. Hg.

W_s : south component of wind velocity in m/sec.

W_w: west 〃 〃 〃 .

1. Introduction.

The Misumi harbor is on the extremity of Uto peninsula and is northern coast of Siranuhi sea. It is often suffered by tidal waves. The meteorological tide at Misumi harbor is synoptic analyzed by taking into account the atmospheric pressure and wind velocity at Kumamoto weather station. The writer left out of consideration the errors which might be introduced from the difference of the geographical and meteorological condition between Misumi and Kumamoto.

2. Methods for Analysis.

Firstly, the writer has deduced the astronomical tide, using the method of interpolation instead of harmonic analysis. In other words, the writer has used the tidal data for 6 days, 3 days before and 3 days after the day of typhoon, and computed the astronomical tide on the day of typhoon by interpolation, taking into account the phase-lag of 50 minutes a day.

Secondly, in regard to the atmospheric pressure, taking the mean values during the seven days as the normal values, the writer calculated the barometric depression (ΔP) for the day of typhoon. The wind velocities are decomposed into south (W_s) and west (W_w) components. As Kumamoto is about 22km from the Misumi harbor (about 16km from the nearest coast of Shiranuhi sea), the variation of wind at Misumi harbor may be a little of that on Kumamoto. The actual difference at both stations is unknown. The data of wind at Kumamoto one hour after is, therefore, tantatively adopted as the wind at Misumi. The writer assumed, the linear equation:

$$\Delta H = \Delta Pa + W_s b + W_w c$$

between these data, and determined the coefficients a, b and c by the method of least squares.

3. Effect of Typhoon Passing on the Westside of Misumi Harbor.

(1) The writer applied the analysis to the typhoon of July 25-26th 1937 (Fig. 1-I) which was born at a point about 500km to the south-west of Rasa island (128° E, 20° N approximately) on 22th, and reached at a point about 100km to the west of Yakushima on 25th and entered into Japan sea on 26th, and its mean speed is estimated to be about 25km/h.

The astronomical tide is shown by dotted line in Fig. 2.
Table 1 gives meteorological and tidal data of this typhoon.
The writer determined the coefficients a, b and c by using the table 1.
The resulting equation is

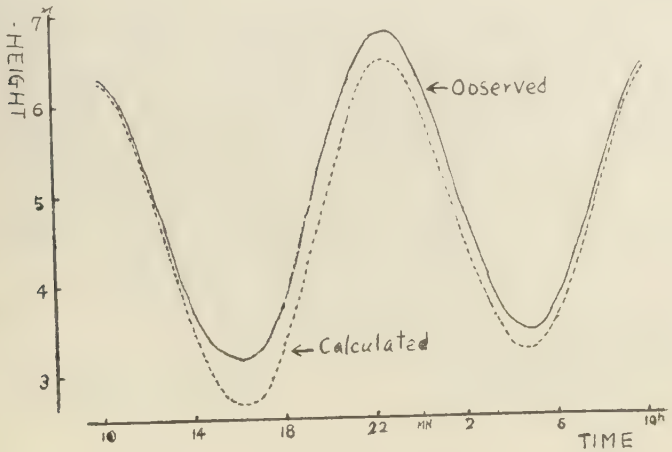


Fig. 2 Tide : Misumi (Jul. 25—26th 1937)

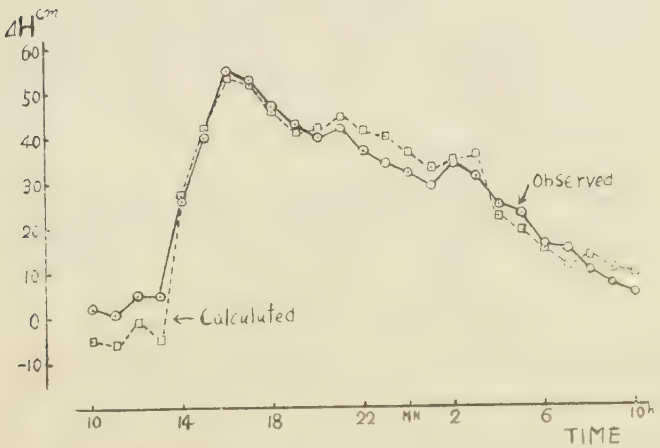


Fig. 3 $\Delta H = 1.39\Delta P + 3.15W_s + 0.68W_w$ (Jul. 25—25th 1937)

$$\Delta H = 1.39\Delta P + 3.15W_s + 0.68W_w$$

where ΔH is in cm., ΔP in mm. Hg. and W_s, W_w in m/s.

Rely on this equation and table 1, the values of ΔH is calculated and represented by dotted line in Fig. 3.

From thisresult, it may be concluded that

(a) the atmospheric pressure acts almost statically,

(b) the effective direction of wind is about 20 degrees to the left of the direction of the axis of the bay. It is probably due to the action of coriolis' force (Fig. 4).

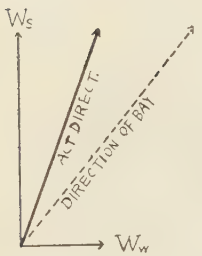


Fig. 4

(2) The equation above obtained is verified by using the data of another typhoon which was born in the East china sea (124° E, 27° N approximately) on 8th, and entered into Japan sea (132° E, 36° N approximately) on Sep. 9th 1934, whose track is shown in Fig. 1-II. The mean speed is estimated to be about 45km/h. The table 2 gives the data of this case.

The writer not considered the timely difference of wind between Misumi and Kumamoto, because, the mean speed of this typhoon is more larger than the before it.

The values of ΔH is calculated by the above equation and is shown by dotted line in Fig. 5.

The result is almost in agreement with observation as shown by Fig. 5.

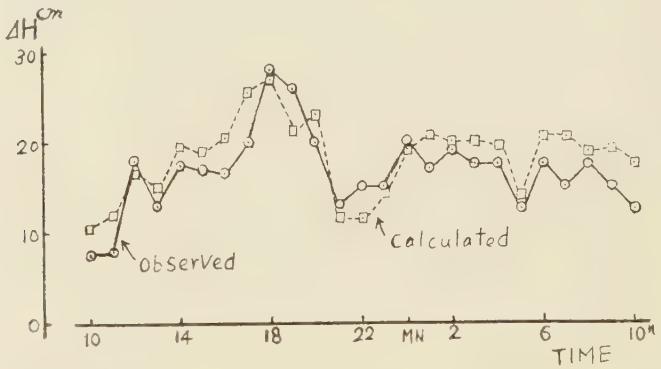


Fig. 5 $\Delta H = 1.39\Delta P + 3.15W_s + 0.68W_w$ (sep. 8—9th 1934)

4. Effect of Typhoon Passing on the Eastside of Misumi Harbor.

The eastern Kyûshû was attacked by Jane (No. 12) typhoon of Sep. 13-14th 1954, whose track is shown in Fig. 1-III. It was estimated that the diameter of "the eye" was about 170-180km and the maximum wind velocity arround it about 60m/s. The mean speed is estimated to be about 30km/h.

To test whether in this case, the equation obtained in the case of typhoon passing on the westside of Misumi harbor is valied or not in this case, the writer has calculated ΔH using it.

The data of wind at Kumamoto one hour after is, therefore, tentatively adopted as

the wind at Misumi.

Table 3 gives meteorological and tidal data of this case. The values of astronomical tide is shown by dotted line in Fig. 6.

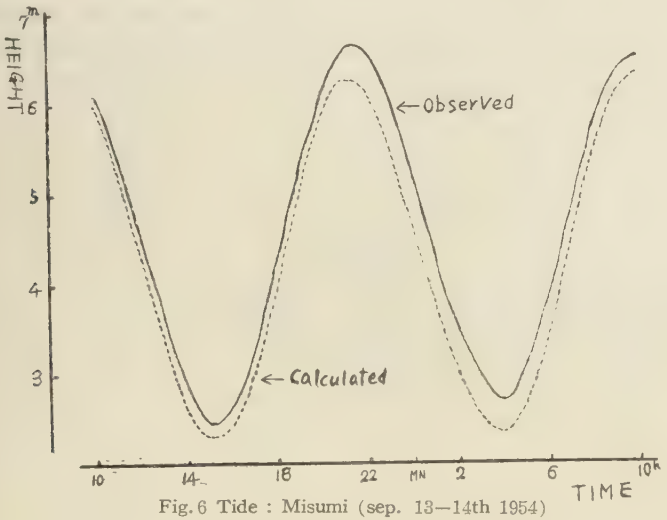


Fig. 6 Tide : Misumi (sep. 13—14th 1954)

The values of ΔH is represented by dotted line in Fig. 7.

The result is almost in agreement with observation, though the calculated values (dotted line in Fig. 7) are slightly higher than the observed values (solid line in Fig. 7).

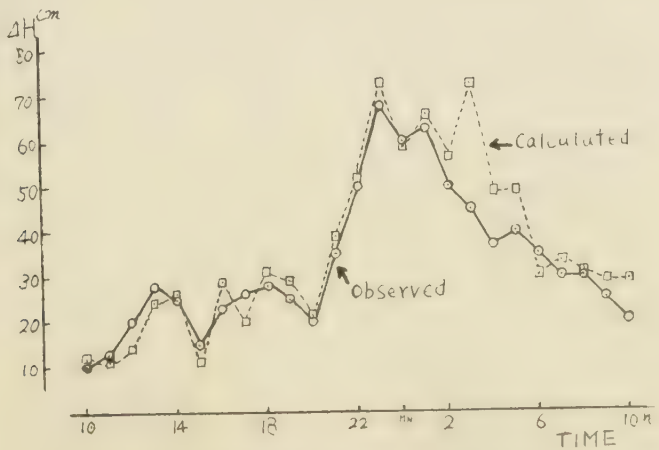


Fig. 7 $\Delta H = 1.39\Delta P + 3.15W_s + 0.68W_m$ (sep. 13—14th 1954)

5. Conclusion.

The equation $\Delta H = 1.39\Delta P + 3.15W_s + 0.68W_m$ is practically good representation of

the meteorological tide due to typhoon at Misumi harbor, whenever typhoon passes on the east or westsiide of the station.

In conclusion, the writer wishes to express his hearty thanks to Mr. I. Kimura (director of Kumamoto weather station) who allowed the writer to use the tidal and meteorological data in the station at the writer disposal, and also to Prof. S. T. Nakamura who gived the theme and kind giudance, and to Prof. T. Murota for his kind encouragement.

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2. 〃 〃 : The investigation of tidal waves on Tôkyô bay (1917)
3. Kumamoto weather station: "Kisyô Saigai-shî".
4. 〃 〃 : "Taihû-shiryo".
5. 〃 〃 : "Taihû (12-gô) Sokuhô".

Table 1. Data (Jul. 25-26th 1937)

	Atm. Press. and Wind (Kumamoto)					Tide (Misumi)	
	700_P+	W	ΔP	W_s	W_w	H	ΔH
TIME	(mm)	(m/s)	(mm)	(m/s)	(m/s)	(cm)	(cm)
10	44.7	ENE 3.8	5.3	−1.4	−3.4	630	2
11	43.8	NE 5.6	6.2	−3.9	−3.9	598	1
12	41.4	NE 5.4	8.6	−3.8	−3.7	530	5
13	39.9	ENE 6.6	10.1	−2.8	−5.9	452	5
14	38.2	ESE 10.6	11.8	−4.2	−9.5	375	26
15	36.4	ESE 15.6	13.6	+6.2	−14.0	334	40
16	35.8	SE 15.5	14.2	+10.9	−10.8	317	55
17	36.6	SSE 13.3	13.4	+12.0	−7.3	329	53
18	38.2	SSE 11.4	11.8	+10.3	−4.6	381	47
19	39.2	S 9.3	10.8	+9.3	0	471	43
20	40.5	S 8.3	9.5	+8.3	0	557	40
21	41.6	S 9.8	8.4	+8.8	0	625	42
22	42.7	S 10.6	7.3	+10.2	0	671	37
23	43.2	S 9.7	6.8	+9.7	0	674	34
0	43.9	S 9.6	6.1	+9.6	0	627	32
1	44.1	S 8.7	5.9	+8.7	0	556	29
2	45.0	SSW 7.8	5.0	+6.9	+3.1	480	43
3	46.0	SSW 8.4	4.0	+7.6	+3.4	408	31
4	47.2	SSW 9.8	2.8	+8.8	+3.8	355	25
5	47.8	SW 6.5	2.2	+4.6	+4.6	342	23
6	48.4	SSW 5.1	1.6	+4.6	+2.0	375	16
7	49.0	SSW 4.1	1.0	+3.8	+1.7	442	15
8	49.5	SW 3.3	0.5	+2.3	+2.3	518	10
9	50.1	SSW 3.9	0	+3.5	+1.6	588	7
10	50.2	SW 3.9	0	+2.7	+2.7	632	5

Table 2. Data (sep. 8-9th 1934)

	Atm. Press. and Wind (Kumamoto)					Tide (Misumi)	
TIME	700_P+ (mm)	W (m/s)	⊿P (m/s)	W_s (m/s)	W_w (m/s)	H (cm)	⊿H (cm)
10	48.3	NNE 0.7	8.7	−0.6	−0.3	580	7
11	48.0	E 0.5	9.0	0	−0.5	510	8
12	47.4	SSE 1.5	9.6	+1.4	−0.6	433	18
13	46.7	SE 0.6	10.3	+0.4	−0.4	362	13
14	45.8	S 1.4	11.2	+1.4	0	309	18
15	45.6	SSW 1.1	11.4	+1.0	+0.4	294	17
16	45.1	SSE 1.8	11.9	+1.6	−0.7	319	17
17	45.1	SSE 3.8	11.9	+3.4	−1.5	387	20
18	44.2	SSE 4.1	12.8	+3.7	−2.0	458	28
19	44.3	SW 1.3	12.7	+0.9	+0.9	566	26
20	44.6	WSW 3.3	12.4	+1.3	+3.3	625	20
21	44.5	NE 2.4	12.5	−1.7	−1.6	642	13
22	43.2	N 2.6	13.8	−2.6	0	610	15
23	41.5	NNW 3.2	15.5	−2.9	+2.0	550	15
0	39.8	NE 1.5	17.2	−1.1	−1.1	480	20
1	38.4	NW 1.9	17.6	−1.4	+1.4	407	17
2	38.5	NE 2.3	17.5	−1.6	−1.6	355	19
3	39.2	NW 2.1	16.8	−1.5	+1.5	335	18
4	41.3	NW 6.5	14.7	−4.7	+4.5	350	17
5	44.2	ESE 1.7	12.8	−0.7	−1.5	405	13
6	45.3	S 1.3	11.7	+1.3	0	488	18
7	46.8	SW 2.4	10.2	+1.7	+1.7	566	15
8	47.5	SSW 2.0	9.5	+1.8	+0.4	627	18
9	48.1	SSE 2.2	8.9	+2.0	−0.9	646	15
10	49.0	S 2.1	8.0	+2.1	0	616	13

Tabe 3. Data (sep. 13-14th 1954)

	Atm. Press. and Wind (Kumamoto)					Tide (Misumi)	
TIME	700_P+ (mm)	W (m/s)	⊿P (mm)	W_s (m/s)	W_w (m/s)	H (cm)	⊿H (cm)
10	32.6	NNE 5.2	18.4	−4.7	−2.1	607	10
11	31.7	NE 5.4	19.3	−3.8	−3.8	537	13
12	30.3	E 8.9	20.7	0	−8.9	450	20
13	29.1	E 10.0	21.9	0	−10.0	366	28
14	27.3	NE 9.1	23.3	−6.4	−6.3	286	25
15	25.5	E 12.4	25.5	0	−12.4	246	15
16	23.7	ENE 11.0	27.3	−4.4	−9.9	264	23
17	22.1	ENE 5.5	28.9	−2.2	−5.0	332	27
18	20.6	ENE 8.7	30.4	−3.5	−7.8	427	28
19	18.7	NE 8.9	32.3	−6.2	−6.2	528	25
20	18.8	E 5.0	32.2	0	−5.0	606	20
21	20.1	S 3.6	30.9	+3.6	0	655	35
22	21.7	SW 13.9	29.3	+9.7	+9.7	658	50
23	24.7	SW 9.6	26.3	+6.7	+6.7	606	68
0	27.3	SW 13.7	23.7	+9.6	+9.6	520	60
1	29.7	SW 11.5	21.3	+8.1	+8.1	432	63
2	32.5	SW 19.3	18.5	+13.5	+13.5	348	50
3	35.4	WSW 16.5	15.6	+6.6	+14.9	294	45
4	36.7	SW 12.4	14.3	+8.7	+8.7	268	37
5	38.8	SW 8.2	12.2	+4.3	+4.3	310	40
6	40.7	SW 7.8	10.3	+5.6	+5.5	384	35
7	41.7	SW 7.3	9.3	+5.1	+5.1	478	30
8	43.1	SW 7.3	7.9	+5.1	+5.1	572	30
9	44.3	SW 8.0	6.9	+5.6	+5.6	630	25
10	45.5	SW 6.5	5.5	+4.6	+4.6	650	20

QUASI-STATIONARY PROCESS (II) CONNECTED SYSTEMS AND LIVING ORGANISM

Shigeichi FUJITA

(Received January 10, 1955)

Abstract

The time variations of forces, fluxes and entropy production of a system when it changes from non-stationary state to stationary state was studied in (I)[1]. For the purpose of promoting the comprehension about the actual process, we have studied the time variations of forces, fluxes and entropy production of a system which is consisted of two connected subsystems. We have studied also the time variations of the free energy of a system and obtained some results and succeeded to give a reasonable explanation about the gradational vicissitudes of living organisms.

1. Introdution

As to the stationary process, Prigogine, de Groot and Meixner treated and gave some conclusions. We considered and studied the same problem and obtained conclusions of different forms and the formula for the time variations of forces, fluxes and entropy production of a system which is changing from non-stationary state to stationary state.

The principal meaning of this study is as follows. When a system changes irreversibly, if we denote the forces, the fluxes and the phenomenological coefficients in a diagonal form by $X_i(i=1,2,\cdots,n)$, J_i and L_i respectively, there is a relation

$$J_i = L_i X_i \ (i=1,2,\cdots,n) \quad \cdots\cdots (1)$$

and the entropy production σ is given by

$$\sigma = \sum_{i=1}^{n} J_i X_i = \sum_{i=1}^{n} L_i X_i^2 \quad \cdots\cdots (2)$$

As the flux J_i occurs in the direction of moderating the force X_i, if the state is not so apart from the stationary state, the time variation of X_i is nearly given by the expression

$$\frac{dX_i}{dt} = \alpha_i (J_{ia} - J_i) \quad \cdots\cdots (3)$$

where J_{ia} is the stationary value of J_i, and α_i is a proportional constant[1]. From this we obtain

$$X_i - X_{ia} = (X_{io} - X_{ia})\, e^{-\alpha_i L_i t} \quad \cdots\cdots (4)$$

$$J_i - J_{ia} = (J_{io} - J_{ia})\, e^{-\alpha_i L_i t} \quad \cdots\cdots (5)$$

where X_{ia} is the force in the stationary state and X_{io} and J_{io} are the initial values of X_i and J_i respectively.

The entropy production σ is given by

$$\sigma = \sum_{i=1}^{n} L_i X_i^2 = \sum_{i=1}^{n} L_i \{X_{ia} + (X_{io} - X_{ia}) e^{-\alpha_i L_i t}\}^2 \quad \cdots\cdots (6)$$

In the above discussion, we take, for simplicity, only one system and assumed the constant effects on the system from the outside systems. The outside systems, however, in actual cases, do not maintain the constant states and continue some variations, and therefore, the effects from outside systems are not to be taken constant.

In this paper, we attempted to study the irreversible process of a system which is under the varying influences from the outside systems. Firstly, we took a system, composed of two connected subsystems, one of which changes under the influence of the other, and studied the time variations of forces, fluxes and entropy productions of the two subsystems.

Also we studied the time variations of the free energy of a system whose stationary state varies gradually, and obtained a result which is supposed to be able to explain some important characters of the living organisms.

2. Connected Two Systems

Systems Ⅰ and Ⅱ are connected, and the forces in the diagonal form in Ⅰ and Ⅱ are denoted by ${}_1X_i$ and ${}_2X_i$ $(i=1, 2, \cdots, n)$ respectively, and the conjugate fluxes in the diagonal form in Ⅰ and Ⅱ are denoted by ${}_1J_i$ and ${}_2J_i$ respectively. It is assumed that the flux ${}_1J_i$ of the system Ⅰ has the effect of increasing the force ${}_2X_i$ of the system Ⅱ. This assumption is acknowledged if we consider some actual connected systems. For example, suppose a system, composed of two isolated equi-potential conductors, if any electric current flows from outside system into one of the conductors, electric potential difference would be produced between the two conductors. Also, suppose a system composed of two gas vessels connected by a narrow tube, if any gas flows from outside system to one of the vessels, the pressure difference would be produced between them and the flow of gas through the narrow tube would be occurred. Many cases, similar to this example, would be easily discovered in the ordinary physical phenomena.

As to the change of forces in the system Ⅰ, from (3) we have the relation

$$\frac{d_1 X_i}{dt} = {}_1\alpha_i (J_{ia} - {}_1J_i) \quad \cdots\cdots (7)$$

where ${}_1\alpha_i$ is the proportional constant between the force ${}_1X_i$ and the conjugate flux ${}_1J_i$ of the system Ⅰ, and J_{ia} is the stationary value of the flux ${}_1J_i$. But as to the changes of forces in the system Ⅱ, we put the next relation (8), using the above assumption, and considering the effect of ${}_2J_i$ upon ${}_2X_i$, like the case of ${}_1J_i$ and ${}_1X_i$

$$\frac{d_2X_i}{dt} = {}_2\alpha_i\,({}_1J_i - {}_2J_i) \quad \cdots\cdots (8)$$

where ${}_2\alpha_i$ is the proportional constant between the force ${}_2X_i$ and the conjugate flux ${}_2J_i$ of the system II. In the stationary state, the fluxes of the two systems becomes equal to J_{ia}, i.e. ${}_2J_i = {}_1J_i = J_{ia}$. If the phenomenological coefficients in a diagonal form of the two systems are denoted by ${}_1L_i$ and ${}_2L_i$, there are the relations

$${}_1L_i\,{}_1X_i = {}_1J_i \quad \cdots\cdots (9)$$

$${}_2L_i\,{}_2X_i = {}_2J_i \quad \cdots\cdots (10)$$

Putting the relations in (7) and (8), we get

$$\frac{1}{{}_1\alpha_i\,{}_1L_i}\frac{d_1J_i}{dt} = (J_{ia} - {}_1J_i) \quad \cdots\cdots (11)$$

$$\frac{1}{{}_2\alpha_i\,{}_2L_i}\frac{d_2J_i}{dt} = ({}_1J_i - {}_2J_i) \quad \cdots\cdots (12)$$

Solving (11), we get

$${}_1J_i - J_{ia} = ({}_1J_{io} - J_{ia})\,e^{-{}_1\beta_i t} \quad \cdots\cdots (13)$$

where ${}_1J_{io}$ is the initial value of ${}_1J_i$ and ${}_1\beta_i = {}_1\alpha_i\,{}_1L_i$.
Putting (9) in (1), we get

$${}_1X_i - {}_1X_{ia} = ({}_1X_{io} - {}_1X_{ia})\,e^{-{}_1\beta_i t} \quad \cdots\cdots (14)$$

where ${}_1X_{ia}$ is the stationary value of ${}_1X_i$, and ${}_1X_{io}$ is the initial value of ${}_1X_i$.

Next, putting the value of ${}_1J_i$ of (13) in (12), we get

$$\frac{1}{{}_2\beta_i}\frac{d_2J_i}{dt} = J_{ia} - {}_2J_i + ({}_1J_{io} - J_{ia})\,e^{-{}_1\beta_i t} \quad \cdots\cdots (15)$$

where ${}_2\beta_i = {}_2\alpha_i\,{}_2L_i$. To solve the equation (15), we put

$${}_2J_i = Ce^{-{}_2\beta_i t} + De^{-{}_1\beta_i t} + J_{ia} \quad \cdots\cdots (16)$$

where C and D are some constants. Putting (16) in (15), we can determine the constant D as the following.

$$D = \frac{({}_1J_{io} - J_{ia})\,{}_2\beta_i}{{}_2\beta_i - {}_1\beta_i} \quad \cdots\cdots (17)$$

The value of C is determined by the initial value ${}_2J_{io}$ of the flux ${}_2J_i$ as follows.

$$C = {}_2J_{io} - J_{ia} - \frac{({}_1J_{io} - J_{ia})\,{}_2\beta_i}{{}_2\beta_i - {}_1\beta_i} \quad \cdots\cdots (18)$$

Therefore, the solution of (15) is obtained as the following

$$ {}_2J_i - J_{ia} = \left\{ {}_2J_{io} - J_{ia} - \frac{(J_{io} - J_{ia})_2\beta_i}{{}_2\beta_i - {}_1\beta_i} \right\} e^{-{}_2\beta_i t} + \frac{({}_1J_{io} - J_{ia})_2\beta_i}{{}_2\beta_i - {}_1\beta_i} e^{-{}_1\beta_i t} $$

or

$$ J_i - J_{ia} = ({}_2J_{io} - J_{ia}) e^{-{}_2\beta_i t} + ({}_1J_{io} - J_{ia}) \frac{{}_2\beta_i}{{}_1\beta_i - {}_2\beta_i} (e^{-{}_2\beta_i t} - e^{-{}_1\beta_i t}) \cdots\cdots (19) $$

Putting (10) in (19), we obtain

$$ {}_2X_i - {}_2X_{ia} = ({}_2X_{io} - {}_2X_{ia}) e^{-{}_2\beta_i t} + \left(\frac{{}_1J_{io}}{{}_2L_i} - {}_2X_{ia} \right) \frac{{}_2\beta_i}{{}_1\beta_i - {}_2\beta_i} (e^{-{}_2\beta_i t} - e^{-{}_1\beta_i t}) $$

..... (20)

The entropy production of these connected systems is expressed by

$$ \sigma = \sum_{i=1}^{n} ({}_1J_i \, {}_1X_i + {}_2J_i \, {}_2X_i) = \sum_{i=1}^{n} ({}_1L_i \, {}_1X_i^2 + {}_2L_i \, {}_2X_i^2) = \sum_{i=1}^{n} \left(\frac{{}_1J_i^2}{{}_1L_i} + \frac{{}_2J_i^2}{{}_2L_i} \right) \cdots (21) $$

where ${}_1J_i$, ${}_1X_i$, ${}_2J_i$ and ${}_2X_i$ are to be expressed by (13), (14), (19) and (20) respectively.

3. Special Cases

(i) The case of ${}_1\beta_i \gg {}_2\beta_i$

In this case ${}_1J_i$ and ${}_1X_i$ attain instantaneously to the stationary values J_{ia} and ${}_1X_{ia}$, and we see from (19) that ${}_2J_i$ changes according as

$$ {}_2J_i - J_{ia} = ({}_2J_{io} - J_{ia}) e^{-{}_2\beta_i t} \cdots\cdots (22) $$

independently of the value of ${}_1J_{io}$ when $t \neq 0$. And from (20) we obtain

$$ {}_2X_i - {}_2X_{ia} = ({}_2X_{io} - {}_2X_{ia}) e^{-{}_2\beta_i t} \cdots\cdots (23) $$

which expresses that ${}_2X_i$ changes independently of the value of ${}_1X_{io}$.

(ii) The case of ${}_1\beta_i \ll {}_2\beta_i$.

In this case ${}_1J_i$ and ${}_1X_i$ change according as (13) and (14). From (19) we obtain

$$ {}_2J_i - J_{ia} = (J_{io} - J_{ia}) e^{-{}_1\beta_i t}, \quad (t \neq 0) \cdots\cdots (24) $$

which expresses that ${}_2J_i$ changes independently of the value of ${}_2J_{io}$. From (20) we obtain

$$ {}_2X_i - {}_2X_{ia} = \left(\frac{{}_1J_{io}}{{}_2L_i} - {}_2X_{ia} \right) e^{-{}_1\beta_i t}, (t \neq 0) \cdots\cdots (25) $$

which expresses that ${}_2X_i$ changes independently of the value of ${}_2X_{io}$.

(iii) The case of ${}_1J_{io} = {}_2J_{io} = 0$ and ${}_1X_{io} = {}_2X_{io} = 0$.

In this case, ${}_1J_i$ and ${}_1X_i$ change according as

$$ {}_1J_i = J_{ia} (1 - e^{-{}_1\beta_i t}) \cdots\cdots (26) $$

and

$$ {}_1X_i = {}_1X_{ia}(1 - e^{-{}_1\beta_i t}) \quad \cdots\cdots (27) $$

${}_2J_i$ changes according as

$$ {}_2J_i - J_{ia} = -J_{ia}e^{-{}_2\beta_i t} - J_{ia}\frac{{}_2\beta_i}{{}_1\beta_i - {}_2\beta_i}(e^{-{}_2\beta_i t} - e^{-{}_1\beta_i t}) $$

or

$$ {}_2J_i = J_{ia}\left(1 - \frac{{}_1\beta_i e^{-{}_2\beta_i t} - {}_2\beta_i e^{-{}_1\beta_i t}}{{}_1\beta_i - {}_2\beta_i}\right) \quad \cdots\cdots (28) $$

and ${}_2X_i$ changes according as

$$ {}_2X_i = {}_2X_{ia}\left(1 - \frac{{}_1\beta_i e^{-{}_2\beta_i t} - {}_2\beta_i e^{-{}_1\beta_i t}}{{}_1\beta_i - {}_2\beta_i}\right) \quad \cdots\cdots (29) $$

(iv) The case of ${}_1J_{io} = {}_2J_{io} = 0$, ${}_1X_{io} = {}_2X_{io} = 0$ and ${}_1\beta_i \fallingdotseq {}_2\beta_i = \beta_i$.

In this case ${}_1J_i$ and ${}_1X_i$ change according as

$$ {}_1J_i = J_{ia}(1 - e^{-\beta_i t}) \quad \cdots\cdots (30) $$

and

$$ {}_1X_i = {}_1X_{ia}(1 - e^{-\beta_i t}) \quad \cdots\cdots (31) $$

respectively, and the relation

$$ {}_1\beta_i e^{-{}_2\beta_i t} - {}_2\beta_i e^{-{}_1\beta_i t} = e^{-{}_2\beta_i t}\{{}_1\beta_i - {}_2\beta_i e^{-({}_1\beta_i - {}_2\beta_i)t}\} $$

$$ \fallingdotseq e^{-{}_2\beta_i t}[{}_1\beta_i - {}_2\beta_i\{1 - ({}_1\beta_i - {}_2\beta_i)t\}] = e^{-{}_2\beta_i t}({}_1\beta_i - {}_2\beta_i)(1 + {}_2\beta_i t) $$

is satisfied, ${}_2J_i$ and ${}_2X_i$ change according as

$$ {}_2J_i = J_{ia}\{1 - (1 + \beta_i t)e^{-\beta_i t}\} \quad \cdots\cdots (32) $$

and

$$ {}_2X_i = {}_2X_{ia}\{1 - (1 + \beta_i t)e^{-\beta_i t}\} \quad \cdots\cdots (33) $$

respectively. Thus we know that ${}_2J_i$ and ${}_2X_i$ approach to the stationary values later than ${}_1J_i$ and ${}_1X_i$. In Fig. 1. the time variations of ${}_1J_i$ and ${}_2J_i$ are shown graphically. In the graph x means $\beta_i t$ and the ordinate means ${}_1J_i/J_{ia}$ and ${}_2J_i/J_{ia}$ respectively.

(v) The case of ${}_1J_{io} = {}_2J_{io} = J_{io}$.

In this case ${}_1J_i$ and ${}_1X_i$ change according as

$$ J_i - J_{ia} = (J_{io} - J_{ia})e^{-{}_1\beta_i t} \quad \cdots\cdots (34) $$

and

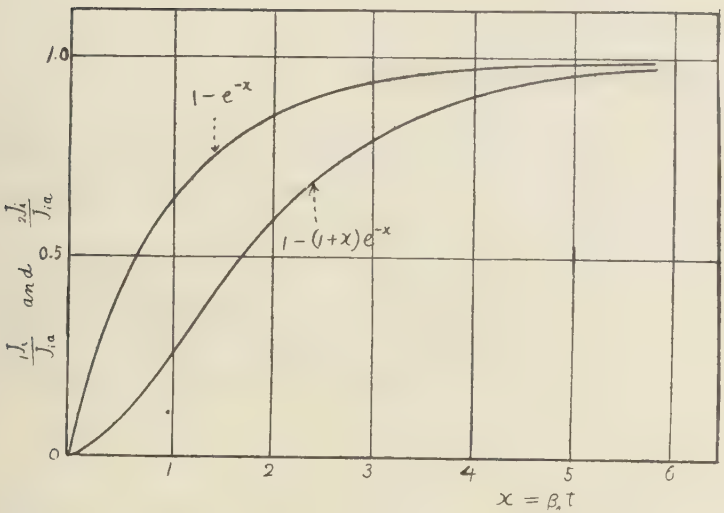

Fig. 1.

$$ {}_1X_i - {}_1X_{ia} = ({}_1X_{io} - {}_1X_{ia})e^{-{}_1\beta_i t},\quad {}_1X_{io} = \frac{J_{io}}{{}_1L_i} \qquad \cdots\cdots (35) $$

respectively. ${}_2J_i$ and ${}_2X_i$ change according as

$$ {}_2J_i - J_{ia} = (J_{io} - J_{ia})\ \frac{{}_1\beta_i e^{-{}_2\beta_i t} - {}_2\beta_i e^{-{}_1\beta_i t}}{{}_1\beta_i - {}_2\beta_i} \qquad \cdots\cdots (36) $$

and

$$ {}_2X_i - {}_2X_{ia} = ({}_2X_{io} - {}_2X_{ia})\ \frac{{}_1\beta_i e^{-{}_2\beta_i t} - {}_2\beta_i e^{-{}_1\beta_i t}}{{}_1\beta_i - {}_2\beta_i},\quad {}_2X_{io} = \frac{J_{io}}{{}_2L_i} \qquad \cdots\cdots (37) $$

respectively.

(vi) The case of ${}_1J_{io} = {}_2J_{io} = J_{io}$ and ${}_1\beta_i = {}_2\beta_i = \beta_i$.

In this case ${}_1J_i$ and ${}_1X_i$ change according as (34) and (35) respectively, if we put β_i for ${}_1\beta_i$. ${}_2J_i$ and ${}_2X_i$ change according as

$$ {}_2J_i - J_{ia} = (J_{io} - J_{ia})\ (1 + \beta_i t)e^{-\beta_i t} \qquad \cdots\cdots (38) $$

and

$$ {}_2X_i - {}_2X_{ia} = ({}_2X_{io} - {}_2X_{ia})\ (1 + \beta_i t)e^{-\beta_i t} \qquad \cdots\cdots (39) $$

respectively.

As we can take J_{io} in the cases (v) and (vi) as a stationary value of flux of the connected systems, the time variations of fluxes and forces in the cases (v) and (vi) can be looked upon as the time variations of those of the systems when the state changes from one

stationary state to another stationary state.

4. Change of Free Energy of an Open System

We suppose an open system A which accepts free energy from the system I and gives free energy to the system II, thus continues irreversible process in the nearly stationary state. It is assumed that the free energy transferred contains the work done by the systems and the free energy transferred by the flow of chemical substances.

If the free energy accepted by the system A from the system I in time dt is denoted by dG_1 and the free energy given by the system A to the system II in the same time is denoted by dG_2 and the entropy increase owing to the irreversible process in the system A is denoted by dS, the increase of the free energy G of the system A is given by

$$dG = dG_1 - dG_2 - TdS \quad \cdots\cdots (40)$$

where T is the temperature of the system.

In the irreversible process in the system A, if we denote the generalized forces by $X_j (j = 1, 2, \cdots\cdots n)$ and the conjugate extensive quantities by x_j, and we put $J_j = \frac{dx_j}{dt}$, then the increase of G per unit time is given by

$$\frac{dG}{dt} = \frac{dG_1}{dt} - \frac{dG_2}{dt} - \sum_{j=1}^{n} X_j J_j \quad \cdots\cdots (41)$$

Now, if we suppose that the system II and the system A are perfectly intercepted and between them there are no interchange of matter and energy, then the free energy of the system A will increase by the supply from the system I, and finally attain to a maximum value G_{a1} and the thermodynamical equilibrium will be formed between the system I and the system A.

Next, if we suppose that the interception between the system II and the system A are released and between them there are interchange of matter and energy, but the system I and the system A are perfectly intercepted and between them there are no interchange of matter and energy, then the free energy of the system A will decrease gradually and finally attain to a minimum value G_{a2} and the thermodynamical equilibrium will be formed between them.

If we assume that the difference of G_{a1} and G_{a2} is small, and the free energy G of the system A, after the interception is released, has the small difference from G_{a1} or G_{a2}, then we can conceive that $\frac{dG_1}{dt}$ is nearly proportional to $G_{a1} - G$, and $\frac{dG_2}{dt}$ is nearly proportional to $G - G_{a2}$, thus we can put

$$\frac{dG_1}{dt} = \alpha'(G_{a1} - G), \quad \frac{dG_2}{dt} = \beta'(G - G_{a2}) \quad \cdots\cdots (42)$$

where α' and β' are constants. We can conceive that the quantity of irreversible process in

the system A is nearly proportional to $G_{a1} - G$ and $G - G_{a2}$, thus we can put

$$T\frac{dS}{dt} = \sum_{j=1}^{n} X_j J_j = \alpha''(G_{a1} - G) + \beta''(G - G_{a2}) \quad \cdots\cdots (43)$$

where α'' and β'' are constants. Accordingly (41) becomes

$$\frac{dG}{dt} = \alpha G_{a1} + \beta G_{a2} - (\alpha + \beta)G \quad \cdots\cdots (44)$$

where

$$\alpha = \alpha' - \alpha'', \quad \beta = \beta' + \beta''.$$

When the system becomes stationary, its free energy has the value

$$G_a = \frac{\alpha G_{a1} + \beta G_{a2}}{\alpha + \beta} \quad \cdots\cdots (45)$$

and the strength of flux of G is

$$\alpha(G_{a1} - G_a) = \beta(G_a - G_{a2}) = \frac{\alpha\beta}{\alpha + \beta}(G_{a1} - G_{a2}) \quad \cdots\cdots (46)$$

We see clearly that this strength becomes small when α or β becomes small. Putting (45) into (44), we get

$$\frac{dG}{dt} = (\alpha + \beta)(G_a - G) \quad \cdots\cdots (47)$$

From this we obtain the solution

$$G = G_a + (G_o - G_a)e^{-(\alpha+\beta)t} \quad \cdots\cdots (48)$$

where G_o is the initial value of G.

If G_{a1} or G_{a2} changes, G_a changes. When the system A attains stationary state and its free energy maintains the value of G_a, if G_{a1} and G_{a2} change to G'_{a1} and G'_{a2} respectively, the stationary value of G changes to

$$G'_a = \frac{\alpha G'_{a1} + \beta G'_{a2}}{\alpha + \beta} \quad \cdots\cdots (49)$$

In this change, as the initial value of G is G_a, G changes according as

$$G = G'_a + (G_a - G'_a)\alpha^{-(\alpha+\beta)t} \quad \cdots\cdots (50)$$

Therefore, if the boundary state keeps stationary, the system A changes according as (48), but if the boundary state changes from one stationary state to another one, the system A changes according as (50).

When G is changing for the stationary value G_a, if the boundary state changes and the

statioary value of G becomes to G'_a, G changes nearly according as

$$G = G'_a + (G' - G'_a)e^{-(\alpha+\beta)t} \quad \cdots\cdots (51)$$

where G' is the value of G at the time when the change of the boundary state occurs.

5. Change of Free Energy of a Living Organism

A living organism will be looked upon as an open system which interchanges matter and energy with surroundings in a quasi-stationary state. The above discussion of 4. might be applied almost nearly to a living organism. It is considered that the fluctuations would occur successively from the body of organism and from the surroundings. Also, when we apply the formula of 4. to a living organism, it is necessary to consider that G_a changes gradually with the growth and the senility of the organism. As the organism grows G_{a1} and G_{a2} increase, and therefore, G_a becomes larger. In the case of senility, G_a becomes smaller. As the area and the property of the surface of the organism would change with its growth, α and β would not maintain constant values. It is almost considered that at the time of vigorous growth, α has some large value, and at the time of senility, α has some small value. As α will have an intimate relation to the assimilation of the organism, α changes with the quantity of assimilation. β will have an intimate relation to the dissimilation of the organism. The work of assimilation would be done by using the free energy of the organism. But the dissimilation would occur naturally, without any work. The work has the effect of promoting dissimilation, but is not absolutely necessary for the dissimilation. After the death of the organism, dissimilation will succeed. Accordingly, it is considered that β would not become so small as α when the organism becomes senile.

Thus it will not be able to express the time variation of G by a simple formula. In a short time interval during which the values of G_a, α and β can be taken as constants, the expressions (47) and (48) may be applied to the organism. In other words, in a short time during which the growth or the senility of the organism can be neglected, the discussion of 4. would be applied. The change of free energy of a living organism would be expressed by (47) or (48) during the interval in which G_a can be taken as a constant, but as G_a is determined by the state of living organism and the state of surroundings, it changes gradually in the long interval. It is considered that the changes of G_a indicate the vicissitudes of the living organism, and the time variation of G expressed by (47) or (48) indicates the fluctuation in a short time interval during which G_a can be taken as a constant.

It is considered that a living organism A is an assembly of the lower class organisms B. For example, a muscle, an internal organ, a bone and others are all assemblies of various cells of enormous number, and a body of some animal is an assembly of muscles, internal organs, bones and others (Bertalanffy's gradational construction of organisms)[2].

If we take A in 4. as the living organism A above mentioned, the change of G_a in the

expression (47) or (48) corresponds to the vicissitudes of the organism A, and the change of G corresponds to the fluctuation of the organism A in a short interval. The fluctuation of the organism A is caused principally from the vicissitudes of the lower grade organisms B. Similarly the fluctuation of B is considered to be caused by the vicissitudes of the lower grade living organisms C.

Thus it is considered that the gradational construction of living organism is expressed by (47) or (48) if we take G_a, α and β as the very slowly varying constants.

If the expression (46) is applied to an organism, it would designate the quantity of metabolism per unit time. It explains the fact that if α or β would become small by senility, sickness and disorder the quantity of metabolism would suddenly decrease.

6. Conclusion

We have studied the time variations of forces, fluxes and entropy production of a system when it changes from non-stationary state to stationary state. And for the purpose of studying practical system which is placed under the effect of slowly changing circumference, we have, firstly, take a system, composed of two subsystems. It is assumed that the fluxes of one of the subsystems have effects upon the forces of the other subsystem. As to the time variations of the forces, the fluxes and the entropy productions of the two subsystems, we have obtained some conclusions.

Next, we have studied time variation of free energy of a system when it changes from non-stationary state to stationary one, and expressed the time variation by a formula. We have also considered the case of a system which changes under the influence of slowly varying surroundings, and applied the result to the growth and the senility of a living organism and have obtained a useful conclusion. The gradational construction of living organism acquired a reasonable explanation by the above conclusion.

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ON A PLANE WAVE PROPAGATED IN AN ELASTIC PLATE WITH INFINITE BREADTH

Ryuzo ADACHI

(Received January 10, 1955)

Abstract

I have treated of a plane wave which is propagated in an elastic plate with infinite breadth along a direction parallel to the surface with constant velocity c satisfying the boundary conditions "*stress*=0 *on the surfaces*".

If the thickness of the plate is $2H$, the number of frequency of the wave is n and let $c_s^2 = \mu/\rho$, $c_p^2 = (\lambda + 2\mu)/\rho$ where ρ is the density of the plate and λ, μ are the **Lamé's** constants, then I have obtained following result.

(i) $c = c_s$ is always possible

(ii) When $c < c_s$ the number of possible value of c is *0* or *1* or *2*, which depends on n, H, c_s, if nH/c_s is very large there is only one value of c and which is equal to that of **Rayleigh's wave.**

(iii) When $c_s < c < c_p$, if nH/c_s is very small, the possible value of c is only one and is equal to $2\sqrt{1-\alpha}\,c_s$, where $\alpha = c_s/c_p$, if nH/c_s is large there can exist many number of c and the numbers depend on nH/c_s

(iv) $c = c_p$ is impossible

(v) When $c_p < c$, if $nH/c_s \leqq \frac{1}{4}$, c can not exist and when $nH/c_s > \frac{1}{4}$ there can exist many number of c and the numbers depend on nH/c_s

1. General theory

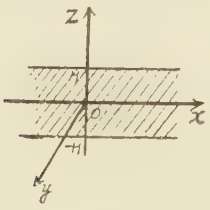

Fig. 1.

In rectangular co-ordinate $o-xyz$, let an elastic plane plate, which is isotropic and uniform, be filled between $z = H$ and $z = -H$ and the direction of a wave propagation be parallel to $x-$ axis. And let n, c be the number of frequency of the wave, velocity of the wave respectively, and u, v, w be the components of the displacement at any point (x, y, z) and put

$$u = Ue^{ik(x-ct)}, \quad v = Ve^{ik(x-ct)}, \quad w = We^{ik(x-ct)} \qquad \cdots\cdots (1)$$

where $\kappa = 2\pi n/c$, t is time, $i = \sqrt{-1}$, and U, V, W are some functions of z only. The fundamental equations are

$$\left.\begin{aligned} &(\lambda+2\mu)\nabla^2\theta=\rho\frac{\partial^2\theta}{\partial t^2}\ , \qquad (\lambda+\mu)\frac{\partial\theta}{\partial x}+\mu\nabla^2 u=\rho\frac{\partial^2 u}{\partial t^2}\\ &(\lambda+\mu)\frac{\partial\theta}{\partial y}+\mu\nabla^2 v=\rho\frac{\partial^2 v}{\partial t^2}\ , \quad (\lambda+\mu)\frac{\partial\theta}{\partial z}+\mu\nabla^2 w=\rho\frac{\partial^2 w}{\partial t^2}\end{aligned}\right\}\cdots(2)$$

where ρ is density of the plate, λ, μ are **Lemé's** constants and

$$\theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\ .$$

Let the boundary conditions be that the stress on $z=\pm H$ is always zero, that is

$$\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=0\ ,\quad \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}=0\ ,\quad \lambda\theta+2\mu\frac{\partial w}{\partial z}=0 \qquad \cdots\cdots(3)$$

when $z=-H$ and $z=H$.

Therefore we must determine U, V, W which satisfy (2) and (3).

2. When $c\neq c_s$ and $c\neq c_p$

Let

$$\left.\begin{aligned} &c_p=\sqrt{\frac{\lambda+2\mu}{\rho}}\ ,\quad c_s=\sqrt{\frac{\mu}{\rho}}\ ,\quad \alpha=\frac{c_s^2}{c_p^2}=\frac{\mu}{\lambda+2\mu}\\ &s=\kappa\sqrt{1-c^2/c_s^2}\ ,\quad r=\kappa\sqrt{1-c^2/c_p^2}\ ,\quad h=\kappa c/c_p\end{aligned}\right\}\cdots\cdots(4)$$

then if $c\neq c_s$ and $c\neq c_p$ we get $sr\neq 0$ and in this case it is easily introduced that the general solutions of (2) in the form (1) are

$$\left.\begin{aligned} u&=\{sMe^{-sz}-sNe^{sz}-\frac{i\kappa}{h^2}(Ae^{rz}+Be^{-rz})\}e^{ik(x-ct)}\\ v&=\{Fe^{-sz}+F'e^{sz}\}e^{ik(x-ct)}\\ w&=\{i\kappa Me^{-sz}+i\kappa Ne^{sz}-\frac{r}{h^2}(Ae^{rz}-Be^{-rz})\}e^{ik(x-ct)}\end{aligned}\right\}\cdots\cdots(5)$$

where M, N, A and B are arbitrary constants.

Accordingly if we put

$$a=-(s^2+\kappa^2)\ ,\quad b=2i\kappa r/h^2\ ,\quad f=2i\kappa\mu s\ ,\quad d=\kappa^2(\rho c^2-2\mu)/h^2 \qquad \cdots\cdots(6)$$

the boundary conditions (3) become

$$Fe^{-Hs}-F'e^{Hs}=0\ ,\quad Fe^{Hs}-F'e^{-Hs}=0 \qquad \cdots\cdots(7)$$

and

$$\left.\begin{aligned} &ae^{-Hs}M + ae^{Hs}N - be^{Hr}A + be^{-Hr}B = 0 \\ &ae^{Hs}M + ae^{-Hs}N - be^{-Hr}A + be^{Hr}B = 0 \\ &-fe^{-Hs}M + fe^{Hs}N + de^{Hr}A + de^{-Hr}B = 0 \\ &-fe^{Hs}M + fe^{-Hs}N + de^{-Hr}A + de^{Hr}B = 0 \end{aligned}\right\} \quad \cdots\cdots (8).$$

At first as $Hs \neq 0$, form (7) we have

$$F = F' = 0 \quad \therefore \ v \equiv 0 \quad \cdots\cdots (9).$$

Next if at least one of M, N, A, B is not zero, from (8) we have

$$\triangle = \begin{vmatrix} ae^{-Hs} & ae^{Hs} & -be^{Hr} & be^{-Hr} \\ ae^{Hs} & ae^{-Hs} & -be^{-Hr} & be^{Hr} \\ -fe^{-Hs} & fe^{Hs} & de^{Hr} & de^{-Hr} \\ -fe^{Hs} & fe^{-Hs} & de^{-Hr} & de^{Hr} \end{vmatrix} = 0 \quad \cdots\cdots (10),$$

expanding this determinant gives us

$$(ad \tanh Hr + bf \tanh Hs)(ad \tanh Hs + bf \tanh Hr) = 0 \quad \cdots\cdots (11)$$

that is

$$ad \tanh Hr + bf \tanh Hs = 0 \quad \cdots\cdots (11)_1$$

or

$$ad \tanh Hs + bf \tanh Hr = 0 \quad \cdots\cdots (11)_2$$

substituting (6) into (11), we have

$$\kappa^2(c^2 - 2c_s^2)^2 \tanh Hr - 4c_s^4 rs \tanh Hs = 0 \quad \cdots\cdots (11)_1'$$

$$\kappa^2(c^2 - 2c_s^2)^2 \tanh Hs - 4c_s^4 rs \tanh Hr = 0 \quad \cdots\cdots (11)_2'$$

From the three equations of (8) we have

$$\begin{aligned} &M : N : A : B = \\ &be^{Hs}[ad(\cosh 2Hr - \cosh 2Hs + \sinh 2Hs) + bf \sinh 2Hr] \\ &: be^{-Hs}[ad(\cosh 2Hs - \cosh 2Hr + \sinh 2Hr) + bf \sinh 2Hr] \\ &: ae^{-Hr}[ad \sinh 2Hs + bf(\cosh 2Hr - \cosh 2Hs + \sinh 2Hs)] \\ &: ae^{Hr}[-ad \sinh 2Hs + bf(\cosh 2Hr - \cosh 2Hs - \sinh 2Hs)] \end{aligned} \quad \cdots\cdots (12),$$

and substituting (11), we get

$$M_1 : N_1 : A_1 : B_1 = b_1 \cosh Hr_1 : b_1 \cosh Hr_1 : a_1 \cosh Hs_1 : -a_1 \cosh Hs_1 \quad \cdots\cdots (13)_1$$

$$M_2 : N_2 : A_2 : B_2 = -b_2 \sinh Hr_2 : b_2 \sinh Hr_2 : a_2 \sinh Hs_2 : a_2 \sinh Hs_2 \quad \cdots\cdots (13)_2.$$

Therefore we have following results,
from (5) and $(13)_1$

$$\left.\begin{aligned} u_1 &= Q_1\left[s_1 b_1\ coshHr_1\ sinhs_1 z + \frac{i\kappa_1 a_1}{h_1^2}\ coshHs_1\ sinhr_1 z\right] e^{i\kappa_1(x-c_1 t)} \\ v_1 &= 0 \\ w_1 &= Q_1\left[-i\kappa_1 b_1\ coshHr_1\ coshs_1 z + \frac{r_1 a_1}{h_1^2}\ coshHs_1\ coshr_1 z\right] e^{i\kappa_1(x-c_1 t)} \end{aligned}\right\} \cdots\cdots (14)_1$$

from (5) and $(13)_2$

$$\left.\begin{aligned} u_2 &= -Q_2\left[s_2 b_2\ sinhHr_2\ coshs_2 z + \frac{i\kappa_2 a_2}{h_2^2}\ sinhHs_2\ coshr_2 z\right] e^{i\kappa_2(x-c_2 t)} \\ v_2 &= 0 \\ w_2 &= Q_2\left[i\kappa_2 b_2\ sinhHr_2\ sinhs_2 z - \frac{r_2 a_2}{h_1^2}\ sinhHs_2\ sinhr_2 z\right] e^{i\kappa_2(x-c_2 t)} \end{aligned}\right\} \cdots\cdots (14)_2$$

where Q_1 , Q_2 are arbitrary constants. Applying (6) we get

$$\left.\begin{aligned} u_1 &= i\kappa_1 P_1\left[2 r_1 s_1\ cosh\ Hr_1\ sinh\ s_1 z - (s_1^2 + \kappa_1^2)\ cosh\ Hs_1\ sinh\ r_1 z\right] e^{i\kappa_1(x-c_1 t)} \\ v_1 &= 0 \\ w_1 &= r_1 P_1\left[2\kappa_1^2\ cosh\ Hr_1\ cosh\ s_1 z - (s_1^2 + \kappa_1^2)\ cosh\ Hs_1\ cosh\ r_1 z\right] e^{i\kappa_1(x-c_1 t)} \end{aligned}\right\} \cdots (15)_1$$

$$\left.\begin{aligned} u_2 &= i\kappa_2 P_2\left[2 r_2 s_2\ sinh\ Hr_2\ cosh\ s_2 z - (s_2^2 + \kappa_2^2)\ sinh\ Hs_2\ cosh\ r_2 z\right] e^{i\kappa_2(x-c_2 t)} \\ v_2 &= 0 \\ w_2 &= r_2 P_2\left[2\kappa_2^2\ sinh\ Hr_2\ sinh\ s_2 z - (s_2^2 + \kappa_2^2)\ sinh\ Hs_2\ sinh\ r_2 z\right] e^{i\kappa_2(x-c_2 t)} \end{aligned}\right\} \cdots (15)_2$$

where P_1 , P_2 are arbitrary constants.

Consequentry, if the equation $(11)_1$ with respect to c has positive root which is not equal to c_s or c_p , the wave $(15)_1$ is possible to exist and if the equation $(11)_2$ has similar root, the wave $(15)_2$ is possible to exist.

3. When $c < c_s$

In this case s, r are real and $0 < s < r$, and $abdf \neq 0$, therefore form (11) we have

$$\frac{tanh\ Hs}{tanh\ Hr} = -\frac{ad}{bf}\ or -\frac{bf}{ad}$$

that is

$$\frac{tanh\ Hs}{tanh\ Hr} = \frac{\kappa^2(c_s^2 - 2c_p^2)^2}{4c_p^4 sr}\ or\ -\frac{4c_p^4 sr}{\kappa^2(c_s^2 - 2c_p^2)^2} \cdots\cdots (16).$$

Putting

$$\eta = c^2/c_s^2\ and\ \varphi(\eta) = (\eta - 2)^2/4\sqrt{(1-\eta)(1-\alpha\eta)} \cdots\cdots (17)$$

(16) becomes

$$\tanh Hs/\tanh Hr = \varphi(\eta) \quad \cdots\cdots (18)_1$$

or

$$\tanh Hs/\tanh Hr = 1/\varphi(\eta) \quad \cdots\cdots (18)_2.$$

The root of $(18)_1$ is the η coordinate of the intersection of two curves $y = \Psi(\eta) = \tanh Hs/\tanh Hr$ and $y=\varphi(\eta)$ in (η, y) plane. Similarly the root of $(18)_2$ is η coordinate of the intersection of two curves $y=\Psi(\eta)$ and $y=1/\varphi(\eta)$. The curves of $y=\varphi(\eta)$ and $y=1/\varphi(\eta)$ are shown in Fig. 2.. The curve of $y = \Psi(\eta)$ is shown in Fig. 3. and if H is very large the curve approaches to DEF and moves to the direction of arrow when H decreases. When $H\to\infty$ the roots of $(18)_1$ and $(18)_2$ coinside with the root of $\varphi(\eta)=/\varphi(\eta)$ and if this root is $\bar{\eta}_0$ and put $c_0^2/c_s^2 = \bar{\eta}_0$ then

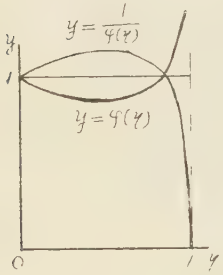

Fig. 2.

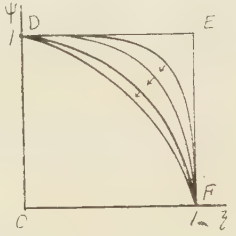

Fig. 3.

$$(\eta_0-2)^2 - 4\sqrt{(1-\eta_0)(1-\alpha\eta_0)} = 0 \quad \cdots\cdots (19)$$

and it shows that c_0 is equal to the velocity of **Rayleigh's wave**. When H is large (not∞) the roots of $(18)_1$ and $(18)_2$ exist each only one and let these be $\bar{\eta}_1$, $\bar{\eta}_2$ then $\bar{\eta}_1<\bar{\eta}_0<\bar{\eta}_2$ and η_1, $\bar{\eta}_2$ are very near to $\bar{\eta}_0$.

If H is very large, we have $Hr \gg Hs$, therefore $\cosh Hs/\cosh Hr \fallingdotseq 0$

and from $(15)_1$ we get

$$\left.\begin{aligned} u_1 &= is_1 R_1 \sinh s_1 z\ e^{i\kappa_1(x-c_1t)} \\ v_1 &= 0 \\ w_1 &= \kappa_1 R_1 \cosh s_1 z\ e^{i\kappa_1(x-c_1t)} \end{aligned}\right\} \quad \cdots\cdots (20)_1$$

where R_1 is arbitrary constant. Similarly from $(15)_2$ we get

$$\left.\begin{aligned} u_2 &= is_2 R_2 \cosh s_2 z\ e^{i\kappa_2(x-c_2t)} \\ v_2 &= 0 \\ w_2 &= \kappa_2 R_2 \sinh s_2 z\ e^{i\kappa_2(x-c_2t)} \end{aligned}\right\} \quad \cdots\cdots (20)_2$$

Therefore if $z = 0$, we have

$$\left.\begin{aligned} u_1 &= v_1 = 0 \\ w_1 &= \kappa_1 R_1\ e^{i\kappa_1(x-c_1t)} \end{aligned}\right\} \quad \cdots\cdots (21)_1$$

$$\left.\begin{aligned} u_2 &= is_2 R_2\ e^{i\kappa_2(x-c_2t)} \\ v_2 &= w_2 = 0 \end{aligned}\right\} \quad \cdots\cdots (21)_2$$

and if z is very large

$$\left.\begin{aligned} U_1 : V_1 : W_1 &= s_1 : 0 : \kappa_1 \\ &= \sqrt{1-\alpha} : 0 : 1 \\ U_2 : V_2 : W_2 &= s_2 : 0 : \kappa_2 \\ &= \sqrt{1-\alpha} : 0 : 1 \end{aligned}\right\} \quad \cdots\cdots (22).$$

Fig. 4. shows when $\alpha = 1/3$, $c_s = 200\ ^m/sec$ $n = 50/sec$ and $H = 0^m$, 1^m, 2^m, 3^m, 4^m, and this indicates that the roots of (18) are follows.

H (m)		0	1	2	3	4
root	η_1	none	0.65	0.79	0.83	0.85
	η_2	none	none	none	0.87	0.86

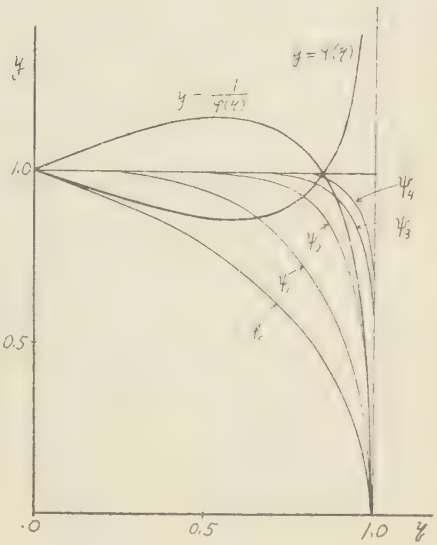

Fig. 4. Ψ_i corresponds to $H = i^m$

4. When $c_s < c < c_p$

In this case r is real and s is imarginary,

and if we put $s_0 = \kappa\sqrt{c^2/c_s^2 - 1}$

$= \dfrac{2\pi n}{c_s}\sqrt{1-\dfrac{1}{\eta}}$, then $s = is_0$, $tanhHs = itanHs_0$, and (11) becomes

$$\left[\kappa^2(\eta-2)^2\, tanhHr + 4rs_0\, tanHs_0\right]\left[4rs_0\, tanhHr - \kappa^2(\eta-2)^2\, tanHs_0\right] = 0 \quad \cdots\cdots (23).$$

Putting

$$\varphi(\eta) = \frac{-(\eta-2)^2}{4\sqrt{(\eta-1)(1-\alpha\eta)}} \quad , \quad 1 < \eta < 1/\alpha \quad \cdots\cdots (24)$$

gives

$$tanHs_0/tanhHr = \varphi(\eta) \quad \cdots\cdots (25)_1$$

or

$$tanHs_0/tanhHr = -1/\varphi(\eta) \quad \cdots\cdots (25)_2.$$

The curves of $y = \varphi(\eta)$ and $y = -1/\varphi(\eta)$ are shown in Fig. 5.. As s_0 is increasing function of η when $1 < \eta$, and when $1 \leqq \eta \leqq \dfrac{1}{\alpha}$, we get

$$0 \leqq s \leqq \frac{2\pi n}{c_s}\sqrt{1-\alpha}$$

therefore, putting $Hs_0 = \frac{\pi}{2} m$ gives

$$0 \leqq m \leqq \frac{4nH}{c_s} \sqrt{1-\alpha} \quad \cdots\cdots\cdots\

1^m, 2^m, 4^m, and the roots of (25) are as follows

$H^{(m)}$		1	2	4
root	$\bar{\eta}_1$	none	none	1.25
	$\bar{\eta}_2$	2.52	none	2.28

If nH is very small, we can consider to be $\frac{4nH}{c_s}\times\sqrt{1-\alpha}<1$ and from (26) we get $q=0$, therefore the equation $\Psi(\eta)=\varphi(\eta)$ has no root, and $\Psi(\eta)=-1/\varphi(\eta)$ has only one root, In this case

$$tanHs_0 \fallingdotseq Hs_0\,,\quad tanhHr \fallingdotseq Hr$$

and from $(25)_2$ we get $\eta=4(1-\alpha)$, that is $c=2\sqrt{1-\alpha}\,c_s$ and let this value be $\bar{c}$, then we have

$$\bar{c}=2\sqrt{1-\alpha}\,c_s \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (29).$$

Substituting $tanHs_0=Hs$, $tanhHr=Hr$ and (29) into $(14)_2$ gives

$$\left.\begin{aligned} u_2 &= \sqrt{3-4\alpha}\,M_2\big[(1-2\alpha)\\ &\times cos\kappa_2\sqrt{3-2\alpha}\,z\\ &+ coshr_2 z\big]e^{i\kappa_2(x-\bar{c}t)}\\ v_2 &= 0\\ w_2 &= -i(1-2\alpha)\,M_2\big[sin\kappa_2\sqrt{3-4\alpha}\,z\\ &+ \sqrt{3-4\alpha}\,sinhr_2 z\big]e^{i\kappa_2(x-\bar{c}t)} \end{aligned}\right\} \quad \cdots (30)$$

where M_2 is constant. Of course Z is small as H is small, we can put

$$cos\kappa_2\sqrt{3-4\alpha}\,z \fallingdotseq 1,\quad coshr_2 z \fallingdotseq 1$$

$$sin\kappa_2\sqrt{3-4\alpha}\,z \fallingdotseq \kappa_2\sqrt{3-4\alpha}\,z,\quad sinhr_2 z \fallingdotseq r_2 z$$

therefore from (30) we get

$$\left.\begin{aligned} u_2 &= N e^{i\kappa_2(x-ct)}\,,\quad v_2=0\\ w_2 &= -i(1-2\alpha)\,N z\, e^{i\kappa_2(x-\bar{c}t)} \fallingdotseq 0 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (30)'$$

where N is constant.

Next, if H is very large, generally $(25)_1$ and $(25)_2$ have many roots and moreover, if z is small, then $cosh\,Hr \gg sinhrz$ and $coshrz$, therefore from $(15)_2$ we get

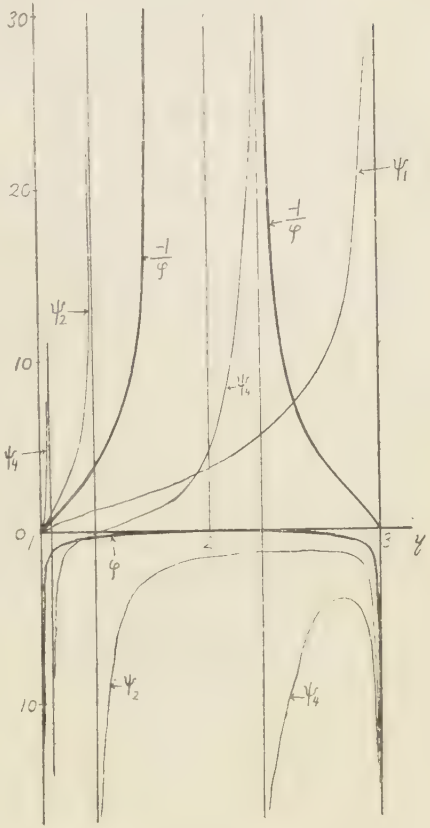

Fig.7. Ψ_1, Ψ_2, Ψ_4 *correspond to* $H=1^m$, 2^m, 4^m *respectively*

$$\left.\begin{aligned} u_1 &= -is_{01} M_1 \sin s_{01} z\, e^{i\kappa_1 (x-c_1 t)} \\ v_1 &= 0 \\ w_1 &= \kappa_1 M_1 \cos s_{01} z\, e^{i\kappa_1 (x-c_1 t)} \end{aligned}\right\} \quad \cdots\cdots (31)_1$$

that is

$$\left.\begin{aligned} u_1 &\doteqdot 0, \quad v_1 = 0 \\ w_1 &\doteqdot N_1 e^{i\kappa_1 (x-c_1 t)} \end{aligned}\right\} \quad \cdots\cdots (31)'_1$$

this is pure transversal wave. Similarly from $(15)_2$ we get

$$\left.\begin{aligned} u_2 &= -s_{02} M_2 \cos s_{02} z\, e^{i\kappa_2 (x-c_2 t)} \\ v_2 &= 0 \\ w_2 &= i\kappa_2 M_2 \sin s_{02} z\, e^{i\kappa_2 (x-c_2 t)} \end{aligned}\right\} \quad \cdots\cdots (31)_2$$

that is

$$\left.\begin{aligned} u_2 &\doteqdot N_2 e^{i\kappa_2 (x-c_2 t)} \\ v_2 &= 0, \quad w_2 \doteqdot 0 \end{aligned}\right\} \quad \cdots\cdots (31)'_2$$

this is pure londitudinal wave.

6. When $c_p < c$

In this case s, r are both imarginary, and putting $s_0 = \kappa\sqrt{\eta-1} = \dfrac{2n\pi}{c_s}\sqrt{1-\dfrac{1}{\eta}}$, $r_0 = \kappa\sqrt{\alpha\eta-1} = \dfrac{2\pi n}{c_s}\sqrt{\alpha-\dfrac{1}{\eta}}$

gives $s=is_0$, $r=ir_0$ where $\eta=c^2/c_s^2>1/\alpha$. Now, if we put

$$\varphi(\eta) = \frac{\kappa^2(\eta-2)^2}{4rs} = \frac{-(\eta-2)^2}{4\sqrt{(\eta-1)(\alpha\eta-1)}} \quad \cdots\cdots (32)$$

$$\Psi(\eta) = \frac{\tanh Hs_0}{\tanh Hr_0} = \tan\frac{2\pi nH}{c_s}\sqrt{1-\frac{1}{\eta}} \Big/ \tan\frac{2\pi nH}{c_s}\sqrt{\alpha-\frac{1}{\eta}} \quad \cdots\cdots (33)$$

then (11) becomes

$$\Psi(\eta) = \varphi(\eta) \quad \cdots\cdots (34)_1$$

$$\Psi(\eta) = 1/\varphi(\eta) \quad \cdots\cdots (34)_2,$$

therefore if these equations have any root, the solution of our problem can exist. At first, the curves of $y=\varphi(\eta)$ and $y=1/\varphi(\eta)$ are shown in **Fig. 8.**

Next, putting

$$\frac{2\pi nH}{c_s}\sqrt{1-\frac{1}{{}_s\eta_{ms}}}=\frac{\pi}{2}m_s,\quad \frac{2\pi nH}{c_s}\sqrt{\alpha-\frac{1}{{}_r\eta_{mr}}}=\frac{\pi}{2}m_r \qquad \cdots\cdots (35)$$

gives

$${}_s\eta_{ms}=\frac{1}{1-K^2m_s^2},\quad {}_r\eta_{mr}=\frac{1}{\alpha-K^2m_r^2} \qquad \cdots\cdots (36)$$

where $K=c_s/4nH$, while $\frac{1}{\alpha}<\eta<\infty$, therefore from (35) we obtain

$$\frac{1}{K}\sqrt{1-\alpha}<m_s<\frac{1}{K},\quad 0<m_r<\frac{1}{K}\sqrt{\alpha}$$

and also $\alpha<1-\alpha<1$ as $\alpha<\frac{1}{2}$, therefore

$$0<m_r<\frac{1}{K}\sqrt{\alpha}<\frac{1}{K}\sqrt{1-\alpha}<m_s<\frac{1}{K} \qquad \cdots\cdots (37).$$

The values of η at which $\Psi(\eta)$ changes its sign are equal to that of when m_r or m_s is equal to integer. If $1/K$ is large, the value of η at which m_r or m_s takes integer exists many number and the intersections of the curve $y=\Psi(\eta)$ and $y=\varphi(\eta)$ or $y=1/\varphi(\eta)$ exist many number, therefore the solutions of our problem can exist many. If $1/K\leqq 1$, then Hs_0 and Hr_0 both less than $\pi/2$, therefore $\Psi(\eta)$ is always positive and (34) has no root and the solution can not exist.

For example, when $\alpha=\frac{1}{3}$, $C_s=200^m/sec$, $n=50/sec$ we have $1/K=Hm^{-1}$ and

(i) when $H=1^m$

$1/K=1$ therefore there is no solution

(ii) when $H=2^m$

$1/K=2$ therefore $\frac{1}{K}\sqrt{1-\alpha}\doteqdot 1.63$, $\frac{1}{K}\sqrt{\alpha}\doteqdot 1.16$ and $0<Hr_0<1.16\times\frac{\pi}{2}<1.63\cdot\frac{\pi}{2}<Hs_0<2\times\frac{\pi}{2}$, ${}_r\eta_1=\frac{1}{\alpha-K^2}=12$ and the curve of $y=\Psi_2(\eta)$ is shown in **Fig.9.**, and $(34)_2$ has only one root $(\eta=3.245)$ and $(34)_1$ has no root.

(iii) when $H=4^m$

$1/K=4$, therefore $\frac{1}{K}\sqrt{1-\alpha}\doteqdot 3.27$, $\frac{1}{K}\sqrt{\alpha}\doteqdot 2.32$ and $0<Hr_0<2.32\times\frac{\pi}{2}<3.27\times\frac{\pi}{2}<Hs_0<4\times\frac{\pi}{2}$, ${}_r\eta_1\doteqdot 3.69$, ${}_r\eta_2=12$. and the curve of $y=\Psi_4(\eta)$ is shown in **Fig. 8.**, **Fig. 9.**, and $(34)_1$, $(34)_2$ has each only one root 3.296, 3.25 respectivery.

5. **When** $C=C_s$ or $C=C_p$

(i) when $C=C_s$

In this case we get $r=K\sqrt{1-\alpha}$, $h=K\sqrt{\alpha}$ and general solutions of (2) in the

forms of (1) are

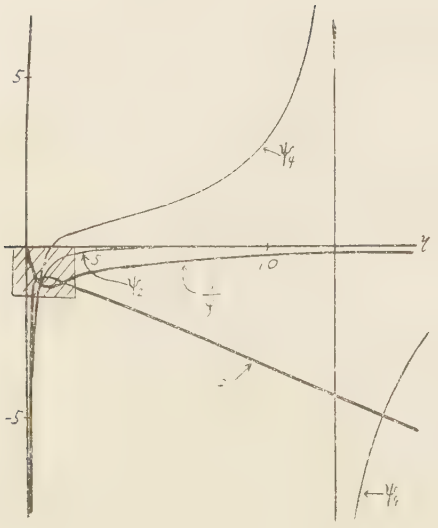

Fig. 8.

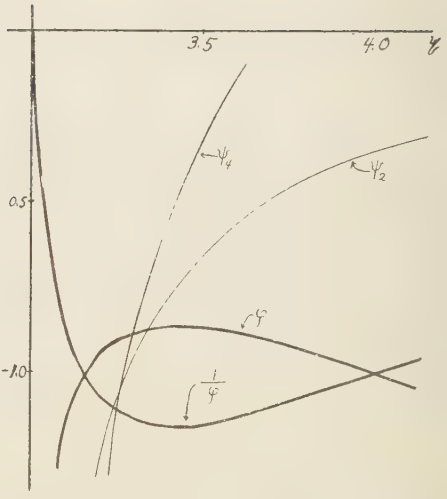

Fig. 9. This figure shows the shaded part in Fig. 8. by magnified scale.

$$\left.\begin{aligned} u &= \left[N - \frac{i\kappa}{h_a^2}\left(Ae^{rz} + Be^{-rz}\right)\right] e^{\,i\kappa(x-c_st)} \\ v &= \left[Fz + F'\right] e^{\,i\kappa(x-c_st)} \\ w &= \left[-i\kappa Nz + Q - \frac{r}{h^2}\left(Ae^{rz} - Be^{-rz}\right)\right] e^{\,i\kappa(x-c_st)} \end{aligned}\right\} \quad \cdots\cdots (38)$$

where N, Q, A, B are arbitrary constants. The boundary conditions (3) become

$$F' = \text{arbitrary, and } F = 0 \quad \cdots\cdots (39)$$

and

$$\left.\begin{aligned} i\kappa Q + \kappa^2 HN - be^{Hr}A + be^{-Hr}B &= 0 \\ i\kappa Q - \kappa^2 HN - be^{-Hr}A + be^{Hr}B &= 0 \\ -\bar{f}\quad N + de^{Hr}A + de^{-Hr}B &= 0 \\ -\bar{f}\quad N + de^{-Hr}A + de^{Hr}B &= 0 \end{aligned}\right\} \quad \cdots\cdots (40)$$

where

$$b = 2i\sqrt{1-\alpha}/\alpha,\ d = -\mu/\alpha,\ \bar{f} = 2i\kappa\mu \quad \cdots\cdots (41).$$

At first, from (39) we have $v = F'e^{\,i\kappa(x-c_st)}$ $\cdots\cdots (42)$,

this means that a transversal wave which displace in the direction of y axis is always possible to exist.

Next, if at least one of Q, N, A, B is not zoro, from (40) we have

$$\begin{vmatrix} i\kappa & \kappa^2 H & -be^{Hr} & be^{-Hr} \\ i\kappa & -\kappa^2 H & -be^{-Hr} & be^{Hr} \\ 0 & -\bar{f} & de^{Hr} & de^{-Hr} \\ 0 & -f & de^{-Hr} & de^{Hr} \end{vmatrix} = 0 \quad \cdots\cdots (43)$$

expanding this determinant, gives

$$\frac{2\pi nH}{C_s} - 4\sqrt{1-\alpha}\tanh\left(\frac{2\pi nH}{C_s}\sqrt{1-\alpha}\right) = 0 \quad \cdots\cdots (44).$$

Generally, this relation is impossible to satisfy, therefore $N = Q = A = B = 0$ and always $u = w = 0$.

(ii) when $C = C_p$

Similarly when $c = c_p$ we can easily prove that any wave can not exist.

6. Conclusion

By above discussion, it is evident that the solutions of the equations (2) under the boundary condition (3) and in the form (1) can exist many numbers when nH is large. But it is yet undecided that actually these waves exist or not. It is evident that if, besides the conditions (3), some conditions e. g initial condition or others are added, the nuumber of solutions decreases. I want to know that these waves experimentally exist or not.

At the end of this paper, I wish to express my thanks to Dr S.T Nakamula for his kind guidance and that he gave me great convenience by lending me many manuscripts.

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APPROXIMATE FORMULAS FOR DEFINITE INTEGRALS AND DIFFERENTIAL COEFFICIENTS

Ryuzo ADACHI

(received January 10, 1955)

I have calculated some approximate formulas which express definite integrals or differential coefficients of a function by terms of several values of the function, and these formulas are very utilizable when we calculate the numerical solutions of some differential equations, integral equations and integro-differential equations.

1. Introduction

Let $f(x)$ be a function which has successive differential coefficients and these are all finite in a given interval (a, b) of x, and put

$$x_i=a+ih,\ \ y_i=f(x_i),\ \ y_i'=f'(x_i)=\frac{d}{dx}f(x_i),\ \ y_i''=f''(x_i)=\frac{d^2}{dx^2}f(x_i)$$

$$F_i=\int_a^{x_i} f(x)\,dx,\ \ i=0,\ 1,\ 2,\ 3,\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $h\neq 0$ and $a\leqq x_i\leqq b$.

In the following, by putting $F_j \fallingdotseq \sum_{i=0}^{p} A_{i,j}\, y_i$

we consider how to determine the coefficients $A_{i,j}$, where j is one of $1, 2, \cdots\cdots\cdots, p$. Similarly, we consider the expressions of y_j', y_j'' by linear forms of y_0, y_1, y_2, $\cdots\cdots$, y_p.

2. Formulas for F_j

Putting

$$F_j=\int_a^{a+ih} f(x)\,dx=\sum_{i=0}^{p} A_{i,j} f(a+ih)-R_{p,j} \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

$$F_j=\sum_{n=1}^{p+1}\frac{j^n h^n}{n!} f^{(n-1)}(a)+\frac{j^{p+2}h^{p+2}}{(p+2)!} f^{(p+1)}(a+j\theta h),\ \ \sum_{j=0}^{p} A_{i,j} f(a+ih)=A_{0,j} f(a)$$

$$+\sum_{i=1}^{p}\left[A_{i,j}\left\{\sum_{m=0}^{p}\frac{i^m h^m}{m!} f^{(m)}(a)+\frac{i^{p+1}h^{p+1}}{(p+1)!} f^{(p+1)}(a+i\theta_i h)\right\}\right]$$

$$0<\theta<1,\ \ 0<\theta_i<1$$

and equating the coefficients of $f(a)$, $f'(a)$, $f''(a)$, $\cdots\cdots\cdots$ in both sides of (1) respectively gives

$$\sum_{i=0}^{p} A_{i,j} = jh, \quad \sum_{i=1}^{p} i^m A_{i,j} = \frac{j^{m+1}}{m+1} h \quad \cdots\cdots (2)$$

$j = one\ of\ 0, 1, 2, \cdots\cdots, p;\ m = 1, 2, 3, \cdots\cdots, p,$

(2) is the simultaneous equations with respect to $A_{0,j}$, $A_{1,j}$, $\cdots\cdots$ $A_{p,j}$ and by solving these equations we get the values of $A_{i,j}$. In this case

$$R_{p,j} = \frac{h^{p+1}}{(p+1)!}\left[\sum_{i=1}^{p} A_{i,j}\, i^{p+1} f^{(p+1)}(a+i\theta_i h) - \frac{h}{p+2} j^{p+2} f^{(p+1)}(a+j\theta h)\right] \cdots\cdots (3).$$

From (2), it is evident that $A_{i,j}$ have h as a factor, then we put $|A_{i,j}| = A_{i,j}|h|$ and let G be the greatest value of $|f^{(p+1)}(x)|$ when $a \leqq x \leqq b$, then we have

$$|R_{p,i}| \leqq \frac{|h^{p+2}|}{(p+1)!}\left[\sum_{i=1}^{p} \bar{A}_{i,j}\, i^{p+1} + \frac{1}{p+2} j^{p+2}\right] G \quad \cdots\cdots (4).$$

Therefore if $|h|$ is very small, we obtain

$$F_j = \sum_{i=0}^{p} A_{i,j}\, y_i + O(h^{p+2}) \quad \cdots\cdots (5).$$

The actual forms of (5) are as follows.

$P=1$ $\quad F_1 = \dfrac{h}{2}(y_0 + y_1) + O(h^3)$

$P=2$ $\quad F_1 = \dfrac{h}{12}(5y_0+8y_1-y_2)+O(h^4)$

$\quad F_2 = \dfrac{h}{3}(y_0+4y_1+y_2)+O(h^5)$ *(1)

$P=3$ $\quad F_1 = \dfrac{h}{24}(9y_0+19y_1-5y_2+y_3)+O(h^5)$

$\quad F_2 = \dfrac{h}{3}(y_0+4y_1+y_2+0y_3)+O(h^5)$ (2)

$\quad F_3 = \dfrac{3h}{8}(y_0+3y_1+3y_2+y_3)+O(h^5)$ (3)

$P=4$ $\quad F_1 = \dfrac{h}{720}(251y_0+646y_1-264y_2+106y_3-19y_4)+O(h^6)$

$\quad F_2 = \dfrac{h}{90}(29y_0+124y_1+24y_2+4y_3-y_4)+O(h^6)$

$\quad F_3 = \dfrac{3h}{80}(9y_0+34y_1+24y_2+14y_3-y_4)+O(h^6)$

$\quad F_4 = \dfrac{2h}{45}(7y_0+32y_1+12y_2+32y_3+7y_4)+O(h^7)$ *

$P=5$ $\quad F_1 = \dfrac{h}{1440}(475y_0+1427y_1-798y_2+482y_3-173y_4+27y_5)+O(h^7)$

$\quad F_2 = \dfrac{h}{90}(28y_0+129y_1+14y_2+14y_3-6y_4+y_5)+O(h^7)$

$$F_3 = \frac{h}{160}(\ 51y_0 + 219y_1 + 114y_2 + 114y_3 - 21y_4 + 3y_5) + O(h^7)$$

$$F_4 = \frac{2h}{45}(\ 7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4 + 0y_5) + O(h^7)$$

$$F_5 = \frac{5h}{288}(\ 19y_0 + 75y_1 + 50y_2 + 50y_3 + 75y_4 + 19y_5) + O(h^7)$$

$P=6$

$$F_1 = \frac{h}{7200}(\ 2272y_0 + 7753y_1 - 5535y_2 + 4470y_3 - 2410y_4 + 753y_5 - 103y_6) + O(h^8)$$

$$F_2 = \frac{h}{3780}(\ 1139y_0 + 5640y_1 + 33y_2 + 1328y_3 - 807y_4 + 264y_5 - 37y_6) + O(h^8)$$

$$F_3 = \frac{h}{2240}(\ 695y_0 + 3240y_1 + 1161y_2 + 2176y_3 - 729y_4 + 216y_5 - 29y_6) + O(h^8)$$

$$F_4 = \frac{h}{945}(-659y_0 + 1392y_1 + 384y_2 + 1504y_3 + 174y_4 + 48y_5 - 8y_6) + O(h^8)$$

$$F_5 = \frac{h}{12096}(\ 3715y_0 + 17400y_1 + 6375y_2 + 16000y_3 + 11625y_4 + 5640y_5 - 275y_6) + O(h^8)$$

$$F_6 = \frac{h}{140}(\ 41y_0 + 216y_1 + 27y_2 + 272y_3 + 27y_4 + 216y_5 + 41y_6) + O(h^9)^*$$

(1) or (2) is the Simpson's 1/3 rule and (3) is 3/8 rule.

3. Formulas for y_j'

Similarly when we determine $B_{i,j}$ from the equations

$$\sum_{i=0}^{p} B_{i,j} = 0\,,\quad \sum_{i=1}^{p} iB_{i,j} = \frac{1}{h}\,,\quad \sum_{i=1}^{p} i^m B_{i,j} = \frac{m}{h} j^{m-1} \qquad \cdots\cdots (6)$$

$j = one\ of\ 0, 1, 2, \cdots\cdots, p;\ m = 2, 3, \cdots\cdots, p,$

we have

$$y_j' = \sum_{i=0}^{p} B_{i,j}\, y_i - R_{p,j} \qquad \cdots\cdots (7)$$

where

$$|R_{p,j}| \leqq \frac{|h^p|}{(p+1)!}\left[\sum_{i=1}^{p} \bar{B}_{i,j}\, i^{p+1} + (p+1)j^p\right] G \qquad \cdots\cdots (8)$$

$$B_{i,j} = |hB_{i,j}|$$

Therefore if $|h|$ is very small, we have

$$y_j' = \sum_{i=0}^{p} B_{i,j}\, y_i + O(h^p) \qquad \cdots\cdots (9).$$

The actual forms of (9) are as follows.

$P=1$ $y_0' = \frac{1}{h}(-y_0+y_1)+O(h),\ y_1' = \frac{1}{h}(-y_0+y_1)+O(h)$

$P=2$ $y_0' = \frac{1}{h}(-\frac{3}{2}y_0+2y_1-\frac{1}{2}y_2)+O(h^2)$

$y_1' = \frac{1}{h}(-\frac{1}{2}y_0+0y_1+\frac{1}{2}y_2)+O(h^2)$

$y_2' = \frac{1}{h}(\frac{1}{2}y_0-2y_1+\frac{3}{2}y_2)+O(h^2)$

$P=3$ $y_0' = \frac{1}{h}(-\frac{11}{6}y_0+3y_1-\frac{3}{2}y_2+\frac{1}{3}y_3)+O(h^3)$

$y_1' = \frac{1}{h}(-\frac{1}{3}y_0-\frac{1}{2}y_1+y_2-\frac{1}{6}y_3)+O(h^3)$

$y_2' = \frac{1}{h}(\frac{1}{6}y_0-y_1+\frac{1}{2}y_2+\frac{1}{3}y_3)+O(h^3)$

$y_3' = \frac{1}{h}(-\frac{1}{3}y_0+\frac{3}{2}y_1-3y_2+\frac{11}{6}y_3)+O(h^3)$

$P=4$ $y_0' = \frac{1}{h}(-\frac{25}{12}y_0+4y_1-3y_2+\frac{4}{3}y_3-\frac{1}{4}y_4)+O(h^4)$

$y_1' = \frac{1}{h}(-\frac{1}{4}y_0-\frac{5}{6}y_1+\frac{3}{2}y_2-\frac{1}{2}y_3+\frac{1}{12}y_4)+O(h^4)$

$y_2' = \frac{1}{h}(\frac{1}{12}y_0-\frac{2}{3}y_1+0y_2+\frac{2}{3}y_3-\frac{1}{12}y_4)+O(h^4)$

$y_3' = \frac{1}{h}(-\frac{1}{12}y_0+\frac{1}{2}y_1-\frac{3}{2}y_2+\frac{5}{6}y_3+\frac{1}{4}y_4)+O(h^4)$

$y_4' = \frac{1}{h}(\frac{1}{4}y_0-\frac{4}{3}y_1+3y_2-4y_3+\frac{25}{12}y_4)+O(h^4)$

$P=5$ $y_0' = \frac{1}{h}(-\frac{137}{60}y_0+5y_1-5y_2+\frac{10}{3}y_3-\frac{5}{4}y_4+\frac{1}{5}y_5)+O(h^5)$

$y_1' = \frac{1}{h}(-\frac{1}{5}y_0-\frac{13}{12}y_1+2y_2-y_3+\frac{1}{3}y_4-\frac{1}{20}y_5)+O(h^5)$

$y_2' = \frac{1}{h}(\frac{1}{20}y_0-\frac{1}{2}y_1-\frac{1}{3}y_2+y_3-\frac{1}{4}y_4+\frac{1}{30}y_5)+O(h^5)$

$y_3' = \frac{1}{h}(-\frac{1}{30}y_0+\frac{1}{4}y_1-y_2+\frac{1}{3}y_3+\frac{1}{2}y_4-\frac{1}{20}y_5)+O(h^5)$

$y_4' = \frac{1}{h}(\frac{1}{20}y_0-\frac{1}{3}y_1+y_2-2y_3+\frac{13}{12}y_4+\frac{1}{5}y_5)+O(h^5)$

$y_5' = \frac{1}{h}(-\frac{1}{5}y_0+\frac{5}{4}y_1-\frac{10}{3}y_2+5y_3-5y_4+\frac{137}{60}y_5)+O(h^5)$

$P=6$ $y_0' = \frac{1}{h}(-\frac{49}{20}y_0+6y_1-\frac{15}{2}y_2+\frac{20}{3}y_3-\frac{15}{4}y_4+\frac{6}{5}y_5-\frac{1}{6}y_6)+O(h^6)$

$y_1' = \frac{1}{h}(-\frac{1}{6}y_0-\frac{77}{60}y_1+\frac{5}{2}y_2-\frac{5}{3}y_3+\frac{5}{6}y_4-\frac{1}{4}y_5+\frac{1}{30}y_6)+O(h^6)$

$$y_2' = \frac{1}{h}\left(-\frac{1}{30}y_0 - \frac{2}{5}y_1 - \frac{7}{12}y_2 + \frac{4}{3}y_3 - \frac{1}{2}y_4 + \frac{2}{15}y_5 - \frac{1}{60}y_6\right) + O(h^6)$$

$$y_3' = \frac{1}{h}\left(-\frac{1}{60}y_0 + \frac{3}{20}y_1 - \frac{3}{4}y_2 + 0\,y_3 + \frac{3}{4}y_4 - \frac{3}{20}y_5 + \frac{1}{60}y_6\right) + O(h^6)$$

$$y_4' = \frac{1}{h}\left(\frac{1}{60}y_0 - \frac{2}{15}y_1 + \frac{1}{2}y_2 - \frac{4}{3}y_3 + \frac{7}{12}y_4 + \frac{2}{5}y_5 - \frac{1}{30}y_6\right) + O(h^6)$$

$$y_5' = \frac{1}{h}\left(-\frac{1}{30}y_0 + \frac{1}{4}y_1 - \frac{5}{6}y_2 + \frac{5}{3}y_3 - \frac{5}{2}y_4 + \frac{77}{60}y_5 + \frac{1}{6}y_6\right) + O(h^6)$$

$$y_6' = \frac{1}{h}\left(\frac{1}{6}y_0 - \frac{6}{5}y_1 + \frac{15}{4}y_2 - \frac{20}{3}y_3 + \frac{15}{2}y_4 - 6\,y_5 + \frac{49}{20}y_6\right) + O(h^6)$$

$P=7$

$$y_0' = \frac{1}{h}\left(-\frac{363}{140}y_0 + 7\,y_1 - \frac{21}{2}y_2 + \frac{35}{3}y_3 - \frac{35}{4}y_4 + \frac{21}{5}y_5 - \frac{7}{6}y_6 + \frac{1}{7}y_7\right) + O(h^7)$$

$$y_1' = \frac{1}{h}\left(-\frac{1}{7}y_0 - \frac{29}{20}y_1 + 3\,y_2 - \frac{5}{2}y_3 + \frac{5}{3}y_4 - \frac{3}{4}y_5 + \frac{1}{5}y_6 - \frac{1}{42}y_7\right) + O(h^7)$$

$$y_2' = \frac{1}{h}\left(\frac{1}{42}y_0 - \frac{1}{3}y_1 - \frac{47}{60}y_2 + \frac{5}{3}y_3 - \frac{5}{6}y_4 + \frac{1}{3}y_5 - \frac{1}{12}y_6 + \frac{1}{105}y_7\right) + O(h^7)$$

$$y_3' = \frac{1}{h}\left(-\frac{1}{105}y_0 + \frac{1}{10}y_1 - \frac{3}{5}y_2 - \frac{1}{4}y_3 + y_4 - \frac{3}{10}y_5 + \frac{1}{15}y_6 - \frac{1}{140}y_7\right) + O(h^7)$$

$$y_4' = \frac{1}{h}\left(\frac{1}{140}y_0 - \frac{1}{15}y_1 + \frac{3}{10}y_2 - y_3 + \frac{1}{4}y_4 + \frac{3}{5}y_5 - \frac{1}{10}y_6 + \frac{1}{105}y_7\right) + O(h^7)$$

$$y_5' = \frac{1}{h}\left(-\frac{1}{105}y_0 + \frac{1}{12}y_1 - \frac{1}{3}y_2 + \frac{5}{6}y_3 - \frac{5}{3}y_4 + \frac{47}{60}y_5 + \frac{1}{3}y_6 - \frac{1}{42}y_7\right) + O(h^7)$$

$$y_6' = \frac{1}{h}\left(\frac{1}{42}y_0 - \frac{1}{5}y_1 + \frac{3}{4}y_2 - \frac{5}{3}y_3 + \frac{5}{2}y_4 - 3\,y_5 + \frac{29}{20}y_6 + \frac{1}{7}y_7\right) + O(h^7)$$

$$y_7' = \frac{1}{h}\left(-\frac{1}{7}y_0 + \frac{7}{6}y_1 - \frac{21}{5}y_2 + \frac{35}{4}y_3 - \frac{35}{3}y_4 + \frac{21}{2}y_5 - 7\,y_6 + \frac{363}{140}y_7\right) + O(h^7)$$

$P=8$

$$y_0' = \frac{1}{h}\left(-\frac{761}{280}y_0 + 8\,y_1 - 14y_2 + \frac{56}{3}y_3 - \frac{35}{2}y_4 + \frac{56}{5}y_5 - \frac{14}{3}y_6 + \frac{8}{7}y_7 - \frac{1}{8}y_8\right) + O(h^8)$$

$$y_1' = \frac{1}{h}\left(-\frac{1}{8}y_0 - \frac{223}{140}y_1 + \frac{7}{2}y_2 - \frac{7}{2}y_3 + \frac{35}{12}y_4 - \frac{7}{4}y_5 + \frac{7}{10}y_6 - \frac{1}{6}y_7 + \frac{1}{56}y_8\right) + O(h^8)$$

$$y_2' = \frac{1}{h}\left(\frac{1}{56}y_0 - \frac{2}{7}y_1 - \frac{19}{20}y_2 + 2\,y_3 - \frac{5}{4}y_4 + \frac{2}{3}y_5 - \frac{1}{4}y_6 + \frac{2}{35}y_7 - \frac{1}{168}y_8\right) + O(h^8)$$

$$y_3' = \frac{1}{h}\left(-\frac{1}{168}y_0 + \frac{1}{14}y_1 - \frac{1}{2}y_2 - \frac{9}{20}y_3 + \frac{5}{4}y_4 - \frac{1}{2}y_5 + \frac{1}{6}y_6 - \frac{1}{28}y_7 + \frac{1}{280}y_8\right) + O(h^8)$$

$$y_4' = \frac{1}{h}\left(\frac{1}{280}y_0 - \frac{4}{105}y_1 + \frac{1}{5}y_2 - \frac{4}{5}y_3 + 0\,y_4 + \frac{4}{5}y_5 - \frac{1}{5}y_6 + \frac{4}{105}y_7 - \frac{1}{280}y_8\right) + O(h^8)$$

$$y_5' = \frac{1}{h}\left(-\frac{1}{280}y_0 + \frac{1}{28}y_1 - \frac{1}{6}y_2 + \frac{1}{2}y_3 - \frac{5}{4}y_4 + \frac{9}{20}y_5 + \frac{1}{2}y_6 - \frac{1}{14}y_7 + \frac{1}{168}y_8\right) + O(h^8)$$

$$y_6' = \frac{1}{h}\left(\frac{1}{168}y_0 - \frac{2}{35}y_1 + \frac{1}{4}y_2 - \frac{2}{3}y_3 + \frac{5}{4}y_4 - 2y_5 + \frac{19}{20}y_6 + \frac{2}{7}y_7 - \frac{1}{56}y_8\right) + O(h^8)$$

$$y_7' = \frac{1}{h}\left(-\frac{1}{56}y_0 + \frac{1}{6}y_1 - \frac{7}{10}y_2 + \frac{7}{4}y_3 - \frac{35}{12}y_4 + \frac{7}{2}y_5 - \frac{7}{2}y_6 + \frac{223}{140}y_7 + \frac{1}{8}y_8\right) + O(h^8)$$

$$y_8' = \frac{1}{h}\left(\frac{1}{8}y_0 - \frac{8}{7}y_1 + \frac{14}{3}y_2 - \frac{56}{5}y_3 + \frac{35}{2}y_4 - \frac{56}{3}y_5 + 14y_6 - 8y_7 + \frac{761}{280}y_8\right) + O(h^8)$$

$P=9$

$$y_0' = \frac{1}{h}\left(-\frac{7129}{2520}y_0 + 9y_1 - 18y_2 + 28y_3 - \frac{63}{2}y_4 + \frac{126}{5}y_5 - 14y_6 + \frac{36}{7}y_7 - \frac{9}{8}y_8 + \frac{1}{9}y_9\right) + O(h^9)$$

$$y_1' = \frac{1}{h}\left(-\frac{1}{9}y_0 - \frac{481}{280}y_1 + 4y_2 - \frac{14}{3}y_3 + \frac{14}{3}y_4 - \frac{7}{2}y_5 + \frac{28}{15}y_6 - \frac{2}{3}y_7 + \frac{1}{7}y_8 - \frac{1}{72}y_9\right) + O(h^9)$$

$$y_2' = \frac{1}{h}\left(\frac{1}{72}y_0 - \frac{1}{4}y_1 - \frac{153}{140}y_2 + \frac{7}{3}y_3 - \frac{7}{4}y_4 + \frac{7}{6}y_5 - \frac{7}{12}y_6 + \frac{1}{5}y_7 - \frac{1}{24}y_8 + \frac{1}{252}y_9\right) + O(h^9)$$

$$y_3' = \frac{1}{h}\left(-\frac{1}{252}y_0 + \frac{3}{56}y_1 - \frac{3}{7}y_2 - \frac{37}{60}y_3 + \frac{3}{2}y_4 - \frac{3}{4}y_5 + \frac{1}{3}y_6 - \frac{3}{28}y_7 + \frac{3}{140}y_8 - \frac{1}{504}y_9\right) + O(h^9)$$

$$y_4' = \frac{1}{h}\left(\frac{1}{504}y_0 - \frac{1}{42}y_1 + \frac{1}{7}y_2 - \frac{2}{3}y_3 - \frac{1}{5}y_4 + y_5 - \frac{1}{3}y_6 + \frac{2}{21}y_7 - \frac{1}{56}y_8 + \frac{1}{630}y_9\right) + O(h^9)$$

$$y_5' = \frac{1}{h}\left(-\frac{1}{630}y_0 + \frac{1}{56}y_1 - \frac{2}{21}y_2 + \frac{1}{3}y_3 - y_4 + \frac{1}{5}y_5 + \frac{2}{3}y_6 - \frac{1}{7}y_7 + \frac{1}{42}y_8 - \frac{1}{504}y_9\right) + O(h^9)$$

$$y_6' = \frac{1}{h}\left(\frac{1}{504}y_0 - \frac{3}{140}y_1 + \frac{3}{28}y_2 - \frac{1}{3}y_3 + \frac{3}{4}y_4 - \frac{3}{2}y_5 + \frac{37}{60}y_6 + \frac{3}{7}y_7 - \frac{3}{56}y_8 + \frac{1}{252}y_9\right) + O(h^9)$$

$$y_7' = \frac{1}{h}\left(-\frac{1}{252}y_0 + \frac{1}{24}y_1 - \frac{1}{5}y_2 + \frac{7}{12}y_3 - \frac{7}{6}y_4 + \frac{7}{4}y_5 - \frac{7}{3}y_6 + \frac{153}{140}y_7 + \frac{1}{4}y_8 - \frac{1}{72}y_9\right) + O(h^9)$$

$$y_8' = \frac{1}{h}\left(\frac{1}{72}y_0 - \frac{1}{7}y_1 + \frac{2}{3}y_2 - \frac{28}{15}y_3 + \frac{7}{2}y_4 - \frac{14}{3}y_5 + \frac{14}{3}y_6 - 4y_7 + \frac{481}{280}y_8 - \frac{1}{9}y_9\right) + O(h^9)$$

$$y_9' = \frac{1}{h}\left(-\frac{1}{9}y_0 + \frac{9}{8}y_1 - \frac{36}{7}y_2 + 14y_3 - \frac{126}{5}y_4 + \frac{63}{2}y_5 - 28y_6 + 18y_7 - 9y_8 + \frac{7129}{2520}y_9\right) + O(h^9)$$

4. Formulas for y_j''

Similarly when we determine $C_{i,j}$ from the equations

$$\sum_{i=0}^{p} C_{i,j} = 0,\ \sum_{i=1}^{p} iC_{i,j} = 0,\ \sum_{i=1}^{p} i^2 C_{i,j} = \frac{2}{h^2},\ \sum_{i=1}^{p} i^m C_{i,j} = \frac{(m-1)m}{h^2} j^{m-2} \quad \cdots\cdots (10)$$

$$j = \textit{one of } 0, 1, 2, \cdots\cdots, p; \quad m = 3, 4, \cdots\cdots, p,$$

we get

$$y_j'' = \sum_{i=0}^{p} C_{i,j} y_i - R_{p,j} \quad \cdots\cdots (11)$$

where

$$|R_{p,j}| \leqq \frac{|h^{p-1}|}{(p+1)!}\left[\sum_{i=1}^{p} i^{p+1} \bar{C}_{i,j} + p(p+1) j^{p-1}\right] G \quad \cdots\cdots (12).$$

$$\bar{C}_{i,j} = |h^2 C_{i,j}|$$

Therefore if $|h|$ is very small, we have

$$y_j'' = \sum_{i=0}^{p} C_{i,j} y_i + O(h^{p-1}) \quad \cdots\cdots (13).$$

The actual forms of (13) are as follows.

$P=2$ $\quad y_0'' = \frac{1}{h^2}(y_0 - 2y_1 + y_2) + O(h)$

$$y_1'' = \frac{1}{h^2}(y_0 - 2y_1 + y_2) + O(h^2)^*$$

$$y_2'' = \frac{1}{h^2}(y_0 - 2y_1 + y_2) + O(h)$$

$P=3$ $\quad y_0'' = \frac{1}{h^2}(2y_0 - 5y_1 + 4y_2 - y_3) + O(h^2)$

$$y_1'' = \frac{1}{h^2}(y_0 - 2y_1 + y_2 + 0y_3) + O(h^2)$$

$$y_2'' = \frac{1}{h^2}(0y_0 + y_1 - 2y_2 + y_3) + O(h^2)$$

$$y_3'' = \frac{1}{h^2}(-y_0 + 4y_1 - 5y_2 + 2y_3) + O(h^2)$$

$P=4$

$$y_0'' = \frac{1}{h^2}\left(\frac{35}{12}y_0 - \frac{26}{3}y_1 + \frac{19}{2}y_2 - \frac{14}{3}y_3 + \frac{11}{12}y_4\right) + O(h^3)$$

$$y_1'' = \frac{1}{h^2}\left(\frac{11}{12}y_0 - \frac{5}{3}y_1 + \frac{1}{2}y_2 + \frac{1}{3}y_3 - \frac{1}{12}y_4\right) + O(h^3)$$

$$y_2'' = \frac{1}{h^2}\left(-\frac{1}{12}y_0 + \frac{4}{3}y_1 - \frac{5}{2}y_2 + \frac{4}{3}y_3 - \frac{1}{12}y_4\right) + O(h^4)^*$$

$$y_3'' = \frac{1}{h^2}\left(-\frac{1}{12}y_0 + \frac{1}{3}y_1 + \frac{1}{2}y_2 - \frac{5}{3}y_3 + \frac{11}{12}y_4\right) + O(h^3)$$

$$y_4'' = \frac{1}{h^2}\left(\frac{11}{12}y_0 - \frac{14}{3}y_1 + \frac{19}{2}y_2 - \frac{26}{3}y_3 + \frac{35}{12}y_4\right) + O(h^3)$$

$P=5$

$$y_0'' = \frac{1}{h^2}\left(\frac{15}{4}y_0 - \frac{77}{6}y_1 + \frac{107}{6}y_2 - 13y_3 + \frac{61}{12}y_4 - \frac{5}{6}y_5\right) + O(h^4)$$

$$y_1'' = \frac{1}{h^2}\left(\frac{5}{6}y_0 - \frac{5}{4}y_1 - \frac{1}{3}y_2 + \frac{7}{6}y_3 - \frac{1}{2}y_4 + \frac{1}{12}y_5\right) + O(h^4)$$

$$y_2'' = \frac{1}{h^2}\left(-\frac{1}{12}y_0 + \frac{4}{3}y_1 - \frac{5}{2}y_2 + \frac{4}{3}y_3 - \frac{1}{12}y_4 + 0\,y_5\right) + O(h^4)$$

$$y_3'' = \frac{1}{h^2}\left(0\,y_0 - \frac{1}{12}y_1 + \frac{4}{3}y_2 - \frac{5}{2}y_3 + \frac{4}{3}y_4 - \frac{1}{12}y_5\right) + O(h^4)$$

$$y_4'' = \frac{1}{h^2}\left(\frac{1}{12}y_0 - \frac{1}{2}y_1 + \frac{7}{6}y_2 - \frac{1}{3}y_3 - \frac{5}{4}y_4 + \frac{5}{6}y_5\right) + O(h^4)$$

$$y_5'' = \frac{1}{h^2}\left(-\frac{5}{6}y_0 + \frac{61}{12}y_1 - 13y_2 + \frac{107}{6}y_3 - \frac{77}{6}y_4 + \frac{15}{4}y_5\right) + O(h^4)$$

$P=6$

$$y_0'' = \frac{1}{h^2}\left(\frac{203}{45}y_0 - \frac{87}{5}y_1 + \frac{117}{4}y_2 - \frac{254}{9}y_3 + \frac{33}{2}y_4 - \frac{27}{5}y_5 + \frac{137}{180}y_6\right) + O(h^5)$$

$$y_1'' = \frac{1}{h^2}\left(\frac{137}{180}y_0 - \frac{49}{60}y_1 - \frac{17}{12}y_2 + \frac{47}{18}y_3 - \frac{19}{12}y_4 + \frac{31}{60}y_5 - \frac{13}{180}y_6\right) + O(h^5)$$

$$y_2'' = \frac{1}{h^2}\left(-\frac{13}{180}y_0 + \frac{19}{15}y_1 - \frac{7}{3}y_2 + \frac{10}{9}y_3 + \frac{1}{12}y_4 - \frac{1}{15}y_5 + \frac{1}{90}y_6\right) + O(h^5)$$

$$y_3'' = \frac{1}{h^2}\left(\frac{1}{90}y_0 - \frac{3}{20}y_1 + \frac{3}{2}y_2 - \frac{49}{18}y_3 + \frac{3}{2}y_4 - \frac{3}{20}y_5 + \frac{1}{90}y_6\right) + O(h^6)^*$$

$$y_4'' = \frac{1}{h^2}\left(\frac{1}{90}y_0 - \frac{1}{15}y_1 + \frac{1}{12}y_2 + \frac{10}{9}y_3 - \frac{7}{3}y_4 + \frac{19}{15}y_5 - \frac{13}{180}y_6\right) + O(h^5)$$

$$y_5'' = \frac{1}{h^2}\left(-\frac{13}{180}y_0 + \frac{31}{60}y_1 - \frac{19}{12}y_2 + \frac{47}{18}y_3 - \frac{17}{12}y_4 + \frac{49}{60}y_5 + \frac{137}{180}y_6\right) + O(h^5)$$

$$y_6'' = \frac{1}{h^2}\left(\frac{137}{180}y_0 - \frac{27}{5}y_1 + \frac{33}{2}y_2 - \frac{254}{9}y_3 + \frac{117}{4}y_4 - \frac{87}{5}y_5 + \frac{203}{45}y_6\right) + O(h^5)$$

$P=7$

$$y_0'' = \frac{1}{h}\left(\frac{469}{90}y_0 - \frac{223}{10}y_1 + \frac{879}{20}y_2 - \frac{949}{18}y_3 + 113y_4 - \frac{201}{10}y_5 + \frac{1019}{180}y_6 - \frac{7}{10}y_7\right) + O(h^6)$$

$$y_1'' = \frac{1}{h}\left(\frac{7}{10}y_0 - \frac{7}{18}y_1 - \frac{27}{10}y_2 + \frac{19}{4}y_3 - \frac{67}{18}y_4 + \frac{9}{5}y_5 - \frac{1}{2}y_6\right)$$

$$+\frac{11}{180}y_7)+O(h^6)$$

$$y_2''=\frac{1}{h^2}\left(-\frac{11}{180}y_0+\frac{107}{90}y_1-\frac{21}{10}y_2+\frac{13}{18}y_3+\frac{17}{36}y_4-\frac{3}{10}y_5+\frac{4}{45}y_6\right.$$
$$\left.-\frac{1}{90}y_7\right)+O(h^6)$$

$$y_3''=\frac{1}{h^2}\left(\frac{1}{90}y_0-\frac{3}{20}y_1+\frac{3}{2}y_2-\frac{49}{18}y_3+\frac{3}{2}y_4-\frac{3}{20}y_5+\frac{1}{90}y_6\right.$$
$$\left.+0\,y_7\right)+O(h^6)$$

$$y_4''=\frac{1}{h^2}\left(0\,y_0+\frac{1}{90}y_1-\frac{3}{20}y_2+\frac{3}{2}y_3-\frac{49}{18}y_4+\frac{3}{2}y_5-\frac{3}{20}y_6\right.$$
$$\left.+\frac{1}{90}y_7\right)+O(h^6)$$

$$y_5''=\frac{1}{h^2}\left(-\frac{1}{90}y_0+\frac{4}{45}y_1-\frac{3}{10}y_2+\frac{17}{36}y_3+\frac{13}{18}y_4-\frac{21}{10}y_5+\frac{107}{90}y_6\right.$$
$$\left.-\frac{11}{180}y_7\right)+O(h^6)$$

$$y_6''=\frac{1}{h^2}\left(\frac{11}{180}y_0-\frac{1}{2}y_1+\frac{9}{5}y_2-\frac{67}{18}y_3+\frac{19}{4}y_4-\frac{27}{10}y_5-\frac{7}{18}y_6\right.$$
$$\left.+\frac{7}{10}y_7\right)+O(h^6)$$

$$y_7''=\frac{1}{h^2}\left(-\frac{7}{10}y_0+\frac{1019}{180}y_1-\frac{201}{10}y_2+41y_3-\frac{949}{18}y_4+\frac{879}{20}y_5-\frac{223}{10}y_6\right.$$
$$\left.+\frac{469}{90}y_7\right)+O(h^6)$$

$P=8$

$$y_0''=\frac{1}{h^2}\left(\frac{29531}{5040}y_0-\frac{962}{35}y_1+\frac{621}{10}y_2-\frac{4006}{45}y_3+\frac{691}{8}y_4-\frac{282}{5}y_5+\frac{2143}{90}y_6\right.$$
$$\left.-\frac{206}{35}y_7+\frac{363}{560}y_8\right)+O(h^7)$$

$$y_1''=\frac{1}{h^2}\left(\frac{363}{560}y_0+\frac{8}{315}y_1-\frac{83}{20}y_2+\frac{153}{20}y_3-\frac{529}{72}y_4+\frac{47}{10}y_5-\frac{39}{20}y_6\right.$$
$$\left.+\frac{599}{1260}y_7-\frac{29}{560}y_8\right)+O(h^7)$$

$$y_2''=\frac{1}{h^2}\left(-\frac{29}{560}y_0+\frac{39}{35}y_1-\frac{331}{180}y_2+\frac{1}{5}y_3+\frac{9}{8}y_4-\frac{37}{45}y_5+\frac{7}{20}y_6\right.$$
$$\left.-\frac{1}{35}y_7+\frac{47}{5040}y_8\right)+O(h^7)$$

$$y_3''=\frac{1}{h^2}\left(\frac{47}{5040}y_0-\frac{19}{140}y_1+\frac{29}{20}y_2-\frac{118}{45}y_3+\frac{11}{8}y_4-\frac{1}{20}y_5-\frac{7}{180}y_6\right.$$
$$\left.+\frac{1}{70}y_7-\frac{1}{560}y_8\right)+O(h^7)$$

$$y_4''=\frac{1}{h^2}\left(-\frac{1}{560}y_0+\frac{8}{315}y_1-\frac{1}{5}y_2+\frac{8}{5}y_3-\frac{205}{72}y_4+\frac{8}{5}y_5-\frac{1}{5}y_6\right.$$
$$\left.+\frac{8}{315}y_7-\frac{1}{560}y_8\right)+O(h^8)^*$$

$$y_5''=\frac{1}{h^2}\left(-\frac{1}{560}y_0+\frac{1}{70}y_1-\frac{7}{180}y_2-\frac{1}{20}y_3+\frac{11}{8}y_4-\frac{118}{45}y_5+\frac{29}{20}y_6\right.$$
$$\left.-\frac{19}{140}y_7+\frac{47}{5040}y_8\right)+O(h^7)$$

$$y_6''=\frac{1}{h^2}\left(\frac{47}{5040}y_0-\frac{1}{35}y_1+\frac{7}{20}y_2-\frac{37}{45}y_3+\frac{9}{8}y_4+\frac{1}{5}y_5-\frac{331}{180}y_6\right.$$
$$\left.+\frac{39}{35}y_7-\frac{29}{560}y_8\right)+O(h^7)$$

$$y_7''=\frac{1}{h^2}\left(-\frac{29}{560}y_0+\frac{599}{1260}y_1-\frac{39}{20}y_2+\frac{47}{10}y_3-\frac{529}{72}y_4+\frac{153}{20}y_5-\frac{83}{20}y_6\right.$$
$$\left.+\frac{8}{315}y_7+\frac{363}{560}y_8\right)+O(h^7)$$

$$y_8''=\frac{1}{h^2}\left(\frac{363}{560}y_0-\frac{206}{35}y_1+\frac{2143}{90}y_2-\frac{282}{5}y_3+\frac{691}{8}y_4-\frac{4006}{45}y_5+\frac{621}{10}y_6\right.$$
$$\left.-\frac{962}{35}y_7+\frac{29531}{5040}y_8\right)+O(h^7)$$

5. Some Properties of $hB_{i,j}$, $h^2C_{i,j}$

If we put

$$hB_{i,j}=b_{i,j} \quad \cdots\cdots (14)$$

then $b_{i,j}$ are independent of h and $b_{0,j}$, $b_{1,j}$, $b_{2,j}$ $\cdots\cdots$ $b_{p,j}$ are the solutions of the simultaneous equations

$$\sum_{i=0}^{p} b_{i,j}=0,\quad \sum_{i=1}^{p} ib_{i,j}=0,\quad \sum_{i=1}^{p} i^m b_{i,j}=mj^{m-1} \quad \cdots\cdots (15)$$

where $j=$ one of 0, 1, 2, $\cdots\cdots$, p; $m=2, 3, \cdots\cdots, p$.
From (15), putting

$$\left.\begin{aligned} &\nabla b_{i,j+1}=b_{i,j+1}-b_{i,j},\quad \nabla^2 b_{i,j+1}=\nabla b_{i,j+1}-\nabla b_{i,j}\\ &\cdots\cdots\\ &\cdots\cdots\quad \nabla^n b_{i,j+1}=\nabla^{n-1}b_{i,j+1}-\nabla^{n-1}b_{i,j}\end{aligned}\right\} \quad \cdots\cdots (16)$$

gives

$$\sum_{i=0}^{p}\nabla^s b_{i,s+t}=0,\quad \sum_{i=1}^{p} i\nabla^s b_{i,s+t}=0 \quad \cdots\cdots (17)$$

$$\sum_{i=1}^{p} i^m\nabla^s b_{i,s+t}=0 \quad \cdots\cdots (18)$$

where $m=2, 3, \cdots\cdots, p$; $s=m, m+1, \cdots\cdots, p$; $t=0, 1, 2, \cdots\cdots, p-s$.
From (17), (18) we have

$$\sum_{i=0}^{p}\nabla^p b_{i,p}=0,\quad \sum_{i=1}^{p} i^m\nabla^p b_{i,p}=0 \quad \cdots\cdots (19)$$

where $m = 1, 2, 3, \cdots\cdots, p$

$$\text{while} \quad \begin{vmatrix} 1 & 2 & \cdots\cdots & p \\ 1^2 & 2^2 & \cdots\cdots & p^2 \\ 1^3 & 2^3 & \cdots\cdots & p^3 \\ \vdots & \vdots & & \\ 1^p & 2^p & \cdots\cdots & p^p \end{vmatrix} \neq 0$$

therefore we get

$$\nabla^p b_{i,p} = 0 \qquad \cdots\cdots \qquad (20)$$

where $i = 0, 1, 2, \cdots\cdots, p$.

Now, if we consider the determinant

$$\angle = \begin{vmatrix} b_{0,0} & b_{1,0} & \cdots\cdots & b_{p,0} \\ b_{0,1} & b_{1,1} & \cdots\cdots & b_{p,1} \\ b_{0,2} & b_{1,2} & \cdots\cdots & b_{p,2} \\ \vdots & \vdots & & \vdots \\ b_{0,p} & b_{1,p} & \cdots\cdots & b_{p,p} \end{vmatrix} \qquad \cdots\cdots \qquad (21)$$

then by (15), $\angle$ is always zero, that is

$$\triangle = 0 \qquad \cdots\cdots \qquad (22).$$

Next, let $f(x, y)$ be a well behaved function and put $y_j = y(x_j)$, $f_j = f(x_j, y_j)$ and consider the determinant

$$\triangle_i = \begin{vmatrix} b_{0,0} & b_{1,0} & \cdots\cdots & b_{i-1,0} & f_0 & b_{i+1,0} & \cdots\cdots & b_{p,0} \\ b_{0,1} & b_{1,1} & \cdots\cdots & b_{i-1,1} & f_1 & b_{i+1,1} & \cdots\cdots & b_{p,1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{0,p} & b_{1,p} & \cdots\cdots & b_{i-1,p} & f_p & b_{i+1,p} & \cdots\cdots & b_{p,p} \end{vmatrix} \qquad \cdots\cdots \qquad (23)$$

then by applying (20), we can easily show

$$\triangle_i = (-1)^{p+i} \nabla^p f_p \begin{vmatrix} b_{0,0} & b_{1,0} & \cdots\cdots & b_{i-1,0} & b_{i+1,0} & \cdots\cdots & b_{p,0} \\ \nabla b_{0,1} & \nabla b_{1,1} & \cdots\cdots & \nabla b_{i-1,1} & \nabla b_{i+1,1} & \cdots\cdots & \nabla b_{p,1} \\ \nabla^2 b_{0,2} & \nabla^2 b_{1,2} & \cdots\cdots & \nabla^2 b_{i-1,2} & \nabla^2 b_{i+1,2} & \cdots\cdots & \nabla^2 b_{p,2} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \nabla^{p-1} b_{0,p-1} & \nabla^{p-1} b_{1,p-1} & \cdots\cdots & \nabla^{p-1} b_{i-1,p-1} & \nabla^{p-1} b_{i+1,p-1} & \cdots\cdots & \nabla^{p-1} b_{p,p-1} \end{vmatrix} \qquad \cdots\cdots \qquad (24)$$

and if $|h|$ is very small, then $\nabla^p f_\mu = O(h^p)$ and as the determinant in the right side of (24) is independent of h, we get

$$\triangle_i = O(h^p) \qquad \cdots\cdots (25).$$

Similarly if we put

$$h^i C_{i,j} = c_{i,j} \qquad \cdots\cdots (26)$$

then $c_{i,j}$ are independent of h and $c_{0,j}, c_{1,j}, c_{2,j}, \cdots\cdots c_{p,j}$ are the solutions of the simultaneous equations

$$\left.\begin{aligned} &\sum_{i=0}^{p} c_{i,j} = 0, \quad \sum_{i=1}^{p} i\, c_{i,j} = 0 \\ &\sum_{i=1}^{p} i^2 c_{i,j} = 0, \quad \sum_{i=1}^{p} i^m c_{i,j} = m\,(m-1)\, j^{m-2} \end{aligned}\right\} \qquad \cdots\cdots (27)$$

where $j = one\ of\ 0, 1, 2, \cdots\cdots, p; \quad m = 3, 4, \cdots\cdots, p,$

and similarly we have

$$\sum_{i=0}^{p} \nabla^{p-1} c_{i,p-1} = 0, \quad \sum_{i=1}^{p} i^m \nabla^{p-1} c_{i,p-1} = 0 \qquad \cdots\cdots (28)$$

$$\sum_{i=0}^{p} \nabla^{p} c_{i,p} = 0, \quad \sum_{i=1}^{p} i^m \nabla^{p} c_{i,p} = 0 \qquad \cdots\cdots (29)$$

where $m = 1, 2, 3, \cdots\cdots, p,$

and from these, we get

$$\nabla^{p-1} c_{i,p-1} = 0, \quad \nabla^{p} c_{i,p} = 0 \qquad \cdots\cdots (30)$$

where $i = 0, 1, 2, \cdots\cdots, p.$

Now, if we consider the determinant

$$\overline{\triangle} = \begin{vmatrix} c_{0,0} & c_{1,0} & c_{2,0} & \cdots\cdots & c_{p,0} \\ c_{0,1} & c_{1,1} & c_{2,1} & \cdots\cdots & c_{p,1} \\ c_{0,2} & c_{1,2} & c_{2,2} & \cdots\cdots & c_{p,2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{0,p} & c_{1,p} & c_{2,p} & \cdots\cdots & c_{p,p} \end{vmatrix} \qquad \cdots\cdots (31)$$

then by (27), $\overline{\triangle}$ is always zero, that is

$$\overline{\triangle} = 0 \qquad \cdots\cdots (32)$$

and if $Q_{i,j}$ is the cofactor of $c_{i,j}$ in $\overline{\triangle}$, then

$$Q_{i,j} = (-1)^{i+j} \begin{vmatrix} c_{0,0} & c_{1,0} & \cdots & c_{i-1,0} & c_{i+1,0} & \cdots & c_{p,0} \\ c_{0,1} & c_{1,1} & \cdots & c_{i-1,1} & c_{i+1,1} & \cdots & c_{p,1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{0,j-1} & c_{1,j-1} & \cdots & c_{i-1,j-1} & c_{i+1,j-1} & \cdots & c_{p,j-1} \\ c_{0,j+1} & c_{1,j+1} & \cdots & c_{i-1,j+1} & c_{i+1,j+1} & \cdots & c_{p,j+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_{0,p} & c_{1,p} & \cdots & c_{i-1,p} & c_{i+1,p} & \cdots & c_{p,p} \end{vmatrix}$$

$$= \frac{\partial \overline{\triangle}}{\partial c_{i,j}} = \begin{vmatrix} c_{0,0} & c_{1,0} & \cdots & c_{i-1,0} & 0 & c_{i+1,0} & \cdots & c_{p,0} \\ c_{0,1} & c_{1,1} & \cdots & c_{i-1,1} & 0 & c_{i+1,1} & \cdots & c_{p,1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{0,j-1} & c_{1,j-1} & \cdots & c_{i-1,j-1} & 0 & c_{i+1,j-1} & \cdots & c_{p,j-1} \\ c_{0,j} & c_{1,j} & \cdots & c_{i-1,j} & 1 & c_{i+1,j} & \cdots & c_{p,j} \\ c_{0,j+1} & c_{1,j+1} & \cdots & c_{i-1,j+1} & 0 & c_{i+1,j+1} & \cdots & c_{p,j+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ c_{0,p} & c_{1,p} & \cdots & c_{i-1,p} & 0 & c_{i+1,p} & \cdots & c_{p,p} \end{vmatrix}$$

and by substituting (30), we get

$$Q_{i,j} = 0 \qquad \cdots\cdots \qquad (33)$$

Similarly, if we consider $\overline{\triangle}_i$ which is made by putting $f_0, f_1, f_2, \cdots\cdots, f_p$ in place of $c_{i,0}, c_{i,1}, \cdots\cdots, c_{i,p}$ in the determinant $\overline{\triangle}$, then $\overline{\triangle}_i$ is always zero, that is

$$\overline{\triangle}_i = 0 \qquad \cdots\cdots \qquad (34)$$

and if $Q_{i,j,k}$ is the determinant which is made by puttung $f_0, f_1, f_2, \cdots\cdots f_{j-1}, f_{j+1}, \cdots\cdots f_p$ in place of $c_{k,0}, c_{k,1}, \cdots\cdots c_{k,j-1}, c_{k,j+1}, \cdots\cdots c_{k,p}$ in $Q_{i,j}$, we get

$$Q_{i,j,k} = O(h^{p-1}) \qquad \cdots\cdots \qquad (35).$$

The equations (22), (25) correspond to that the general solution of first order differential equation has one arbitrary constant and the equations (32), (33), (34), (35) correspond to that the general solution of second order differential equation has two arbitrary constants.

6. Comments

Specially if $f(x)$ is a polynomial of p^{th} order, we have $f(x)^{(p+1)} \equiv 0$ and $G = 0$, therefore $R_{p,j} = 0$ in any case of (4), (8) and (12), this means that above formulas indicate that F_j, y_j', y_j'' **are approximated by the corresponding values of the curves of** p^{th}

order which passes $p+1$ points (x_0, y_0), (x_1, y_1), (x_2, y_2), ... (x_p, y_p).

The formulas for y_j' have h^{-1} as a factor, therefore if we wish to obtain y_j' with errors within $O(h^n)$, we must calculate y_i with errors within $O(h^{n+1})$. Similarly for y_j'', y_i with errors within $O(h^{n+2})$.

Above formulas (5), (9) and (13) are very utilizable not only when we calculate definite integrals or differential coefficients but also when we calculate numerical solutions of

"Differential equations which are in type of

$$\frac{dy}{dx} = f(x, y) \cdots\cdots\cdots\cdots \text{condition} \quad \varphi(x, y) = 0 \text{ when } x = a \text{ etc.}$$

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \cdots\cdots \text{conditions} \begin{cases} \varphi(x, y, y') = 0 \text{ when } x = a \\ \Psi(x, y, y') = 0 \text{ when } x = b \end{cases} \text{etc..}$$

Some of integro-differential equations which are in type of

$$\varphi(x, y, y', y'') = \int_a^b f\left(x, \xi, y(\xi), y'(\xi), y''(\xi)\right) d\xi$$

etc. "

ON THE FORM OF A SURFACE OF LIQUID WHICH IS IN EQUILIBRIUM UNDER PRESSURE AND SURFACE-TENSION

Ryuzo ADACHI

(Received January 10, 1955)

1. General Formula

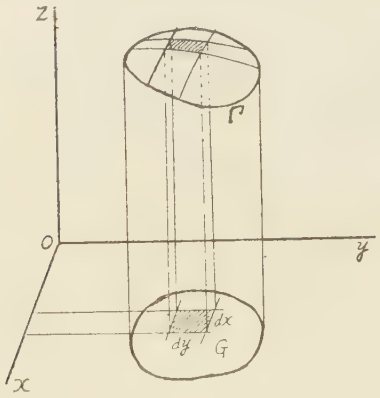

Fig. 1.

Let the equation of a surface of liquid which is in equilibrium under pressure and surfacetension be $z=z(x,y)$ with boundary Γ, then the potential energy J of this surface is

$$J = T\iint_G F dxdy \quad \cdots\cdots (1)$$

where
$$\left.\begin{aligned} F &= \sqrt{1+z_x^2+z_y^2} - \int_0^z \alpha dz, \\ \alpha &= \frac{1}{T}(p-p_0) \end{aligned}\right\} \cdots (2)$$

and p, p_0, T are inside pressure, outside pressure, and surface-tension respectively.

In the state of equilibrium, J is minimum and from the **Euler's** equation

$$\frac{\partial}{\partial x}F_{z_x} + \frac{\partial}{\partial y}F_{z_y} - F_z = 0$$

we have

$$z_{xx}(1+z_y^2) - 2z_{xy}z_x z_y + z_{yy}(1+z_x^2) + \alpha(1+z_x^2+z_y^2)^{3/2} = 0 \quad \cdots\cdots (3),$$

this is the differential equation of the surface. Accordingly, the solution of (3) under given boundary conditions is the required equation of the surface. where α is a function of x, y, z.

Generally we can not solve (3) analytically, and in the following we discuss some special cases.

2. When $p = p_0$

In this case α is zero and (3) becomes

$$z_{xx}(1+z_y^2) - 2z_{xy}z_x z_y + z_{yy}(1+z_x^2) = 0 \quad \cdots\cdots (4)$$

and this is the differential equation of **minimal surface**.

3. When surface of Revolution

When the surface is a surface of revolution, let its equation be

$$y^2 + z^2 = f(x) \quad \cdots\cdots\cdots\

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THE GRAVITATIONAL POTENTIAL ENERGY AND LIFETIME OF A PARTIALLY DEGENERATE STANDARD STELLAR MODEL

Keisuke KAMINISI

(Received January 5, 1955)

Abstract

The gravitational potential energy of a partially degenerate standard stellar model is calculated by making use of the results given by G. W. Wares, and the present results make it possible to estimate the lifetime of such stars. The present work suggests that the main part of the evolutionary path of a white dwarf in the Hertzsprung-Russell diagram lies to the left side of the vertical line representing the spectral type of A_0.

1. Introduction

According to the modern theory of stellar evolution, it is believed that any star, which has nuclear source and belongs to the main-sequence star, shifts leftward along the main-sequence line in the Hertzsprung-Russell diagram in its course of evolution*. The star, which possesses no appreciable mixing current in its interior during the early stage of evolution, will have a fuel-exhausted core of partially degenerate matter and evolve into a red giant [1]. On the other hand the star which possesses the strong mixing currents will have no such core but will contract its radius according to the Kelvin-Helmholtz theory after burning out its fuel. In this case, the energy radiated out of its surface is supplied by a fraction of the decreasing potential energy of the star, the rest of which is reserved as the internal energy for making up quasi-hydrostatic balance in the interior. If the star has a mass larger than Chandrasekhar' slimit $\mathfrak{M} = 5.75 M_\odot$, it expells the excess mass, and after its density becomes considerably high, degeneracy sets in at its center. In this way the star settles down into a white dwarf state after passing through the partially degenerate state [2], [3]. (The "Eddington white dwarf" is considered to perform its condensation without the Bethe cycle or the proton-pron reaction [4], [5].)

In this paper, by making use of the results given by G. W. Wares for the partially degenerate standard model $\mathfrak{M} = 5.6$, we calculate the gravitational potential energy of such a model for each stage of degeneracy. From this result we estimate the time interval between the onset of degeneracy and the final stage of complete degeneracy, assuming that the K-H contraction predominates in this interval. Applying this result we will discuss the lifetime of the stellar evolution and evolutionary paths of the white dwarf in the H-R diagram.

*In the Soviet Union, many scholars, such as A. G. Masewich, believe that stars shift rightward expelling their matter in the outer space.

2. A Partially Degenerate Standard Model [6]

The problem of the structure of partially degenerate standard stellar model was solved by G. W. Wares. According to his theory, the equation of state of the partially degenerate standard model is

$$\left.\begin{aligned} \rho &= B_2 \frac{2}{3} F_{3/2}(\psi) F_{1/2}(\psi) \equiv |B_2| \mu_e \frac{1-\beta}{\beta} \frac{2}{3} F_{2/3}(\psi) F_{1/2}(\psi), \\ P &= A_2 \left[\frac{2}{3} F_{3/2}(\psi) \right]^{5/3} \equiv |A_2| \left(\frac{1-\beta}{\beta} \right)^{5/3} \left[\frac{2}{3} F_{2/3}(\psi) \right]^{5/3}, \end{aligned} \right\} \quad \cdots\cdots (1)$$

where $|B_2| = 2.76 \times 10^6 gm/cm^3$, $|A_2| = 1.003 \times 10^{24} dyne/cm^2$, $\cdots\cdots$ (2)

$\beta = P_g/P$, a constant by the definition of standard model, and both $F_{1/2}(\psi)$ and $F_{3/2}(\psi)$ are the Fermi-Dirac functions defined by

$$F_v = \int_0^\infty \frac{u^v du}{e^{-\psi+u} + 1}$$

The fundamental differential equation for the model is

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left\{ \xi^2 \left[\frac{2}{3} F_{3/2}(\psi) \right]^{2/3} \frac{d\psi}{d\xi} \right\} = -\frac{2}{3} F_{3/2}(\psi) F_{1/2}(\psi), \quad \cdots\cdots (3)$$

where $r = \alpha_2 \xi$, $\cdots\cdots$ (4)

$$\alpha_2 = \left[\frac{2A_2}{3\pi G B_2^2} \right]^{1/2} = \frac{|\alpha_2|}{\mu_e |\beta^2(1-\beta)|^{1/6}} = \frac{0.00944 R_\odot}{\mu_e [\beta^2(1-\beta)]^{1/6}} \quad \cdots\cdots (5)$$

The solution, $\psi(\xi)$, of this equation was tabulated in his paper together with the values of $\rho \propto \frac{2}{3} F_{3/2} F_{1/2}$ and $M \propto -\left(\frac{2}{3} F_{3/2} \right)^{2/3} \xi^2 \psi'(\xi)$ under the following boundary conditions :

at the center $(\xi = 0)$: $\psi(0) = \psi_0$, $\left(\frac{d\psi_0}{d\xi} \right)_0 = 0$,

at the boundary $(\xi = \xi_1)$: $\psi(\xi_1) = -\infty$

By using equations (1), (3) and (4), the mass enclosed in a sphere of radius ξ can be written in the following form :

$$\begin{aligned} M(\xi) &= 4\pi \alpha_2^3 \int_0^\xi \rho \xi^2 d\xi, \\ &= -|C_2| \mu_e^{-2} \left(\frac{1-\beta}{\beta^4} \right)^{1/2} \left(\frac{2}{3} F_{3/2}(\psi) \right)^{2/3} \xi^2 \frac{d\psi}{d\xi} , \end{aligned} \quad \cdots\cdots (6)$$

where $|C_2| = 4\pi |\alpha_2|^3 |B_2| = 9.67 \times 10^{33} gr. = 4.87 M_\odot$.

3. Gravitational Potential Energy

The total negative potential energy, Ω, of the star is [7]

$$\Omega = G\int_0^M \frac{M_r dM_r}{r}, \quad = 4\pi G\int_0^R M_r r\rho(r)dr. \quad \cdots\cdots (7)$$

Substituting Eqs. (1), (4) and (6) into Eq. (7), we find

$$\Omega = G|C_2|^2|\alpha_2|^{-1}\frac{(1-\beta)^{7/6}}{\beta^{11/3}}I,$$

$$= 251\times 10G\frac{M_\odot^2}{R_\odot}\mu_e^{-3}\frac{(1-\beta)^{7/6}}{\beta^{11/3}}I, \quad \cdots\cdots (8)$$

where $I = \int_0^{\xi_1}\left[-\left\{\frac{2}{3}F_{3/2}(\psi)\right\}^{2/3}\xi^2\frac{d\psi}{d\xi}\right]\xi\frac{2}{3}F_{3/2}(\psi)F_{1/2}(\psi)d\xi.$

The numerical interations of I are carried out by making use of table (1) given by Wares for the three values $\psi_0 = 0$, 2 and 5. The values of $(1-\beta)$ in Eq. (8) are given in his table (3) corresponding to the values of $\mathfrak{M} = \frac{M\mu_e^2}{M_\odot}$ and ψ_0. Hence we can easily obtain the desired gravitational energy for the partially degenerate model.

Next the negative potential energy, Ω_∞, in the case of complete degeneracy ($\psi_0 = \infty$ or index of polytrope $n = \frac{3}{2}$) can be calculated by the following equation [8]:

$$\Omega_\infty = \frac{6}{7}G\frac{M^2}{R} = \frac{6}{7}\frac{\mathfrak{M}}{\mathfrak{R}}G\frac{M_\odot^2}{R_\odot}\mu_e^{-3},$$

where values of $\mathfrak{R} = \frac{R\mu_e}{R_\odot}$ are given by Wares' table (3).

The required results are all tabulated in the following table (A).

TABLE (A)

The value of Ω^*, which is defined by $\Omega^* = \Omega\Big/G\frac{M_\odot^2}{R_\odot\mu_e^2}$

$\mathfrak{M}$ $\frac{M\mu_e^2}{M_\odot}$	$\psi_0=\infty$	$\psi_0=5$	$\psi_0=2$	$\psi_0=0$
0.1	0.0894	0.0807	0.0585	0.0252
0.2	0.500	0.407	0.270	0.127
0.3	1.29	1.05	0.684	0.327
0.4	2.52	2.05	1.36	0.640
0.5	4.24	3.45	2.29	1.08
0.6	6.50	5.28	3.50	1.65
0.7	9.26	7.56	5.02	2.37
0.8	12.7	10.3	6.87	3.24
0.9	16.7	13.6	9.04	4.27
1.0	21.3	17.4	11.6	5.47
1.1	26.7	21.7	14.5	6.84
1.2	32.7	26.6	17.7	8.39
1.3	39.4	32.1	21.4	10.1
1.4	46.9	38.2	25.4	12.1
1.5	55.1	44.8	29.9	14.2
1.6	63.9	52.1	34.8	16.6

4. Lifetime of Evolution

If the total energy of a star can be estimated at every stage of its evolution, we can at once guess the lifetime of the star. The total energy of a star consists in the fuel-exhausted system of gravitational, electrostatic, and thermal energy. By making use of the results given in section 3 for the gravitational energy, we are going to estimate the lifetime of the partially degenerate standard model.

As the first approximation we neglect the electrostatic potential energy, for there is good reason to assume that the electrostatic energy is not larger than the gravitational energy and the calculation of the gravitional energy given in the last section has been based on the theory of Wares who tacitly neglected the effect of electrostatic interaction. As for the thermal energy, K, we simply assume the relation : $K = \frac{1}{2}\Omega$.

The above relation is perfectly accurate if the model is in the completely degenerate state, and may be assumed to remain valid as an approximation for the partially degenerate state. At any rate, such an approximation would not cause any serious change in our final conclusion on the lifetime of the partially degenerate standard model.
Hence the total energy E can be written as follows :

$$E = K + (-\Omega) = -\frac{1}{2}\Omega,$$

and consequently

$$\Delta E = -\frac{1}{2}\Delta\Omega. \qquad \cdots\cdots (9)$$

Since this difference is to be equal to the total amount of the radiated energy in this period, we can easily estimate the time length of the period by assuming the value of radiation energy per unit time, that is, the luminosity.

Now the rate of decreasing of energy is

$$\frac{dE}{dt} = -L$$

where L is the luminosity. Hence, by Eq. (9), the time interval between the initial and final stage is given by

$$\int_{ti}^{tf} dt = -\int_{Ei}^{Ef}\frac{dE}{L} = -\frac{1}{L_{if}}\int_{Ei}^{Ef} dE = \frac{\Omega_f - \Omega_i}{2L_{if}},$$

where $L_{if} = \int_{Ei}^{Ef} dE \Big/ \int_{Ei}^{Ef}\frac{dE}{L}$.

Now the exrpession for the luminosity is given by

$$L = 4\pi R^2 \sigma T_s^4 = 3.49\times 10^{34}\Re^2\mu_e^{-2}T_c^{*4} erg/sec.,$$

where $T_e^* = T_e/10000^0 K$.

It is rather tedious to evaluate L_{if} precisely, so we assume that L_{if} may be simply replaced by $\tilde{L}_{if}$ expressed in the following equation :

$$\tilde{L}_{if} = 3.49\times 10^{34}\bar{\mathfrak{R}}^2\mu_e^{-2}(\bar{T}_e^*)^4 \; erg/sec.,$$

where $\bar{\mathfrak{R}} = \frac{1}{2}(\mathfrak{R}_i + \mathfrak{R}_f)$ and $\bar{T}_e^* = \frac{1}{2}(T_{ei}^* + T_{ef}^*)$.

Then we can reach the final relation :

$$t_f - t_i = 1.71\times 10^6 \left(\frac{\Omega_f^*}{\bar{\mathfrak{R}}^2} - \frac{^*\Omega_i}{\bar{\mathfrak{R}}^2}\right)\frac{1}{\mu_e(\bar{T}_e^*)^4} \text{ years.}$$

$\mathfrak{R}$ is given in Wares' table (3) for the given values of ψ_0 and $\mathfrak{M}$. Hence we will get the following table (B), by using table (A).

TABLE (B)

The value of $t_f^* - t_i^*$, reduced lifetime, where $t_k^* = t_k\mu_e(\bar{T}_e^*)^4\times 10^{-9}$ year^{-1} and suffix denotes the value of ψ_0, the degree of degeneracy.

$\mathfrak{M}$	$t_\infty^*-t_5^*$	$t_5^*-t_2^*$	$t_2^*-t_0^*$
0.4	0.18	0.10	0.021
0.8	1.5	0.79	0.17
1.0	2.8	1.5	0.33
1.2	5.0	2.7	0.56
1.6	11.6	6.3	1.3

In table (B), the undetermined T_e^* and μ_e can be estimated approximately from the given spectral type, and from the hydrogen content respectively.

5. Applications

In this section, we assume that $\bar{T}_e^*$ is constant during the K-H contraction as the first approximation. This assumption means that the path of the star in the H-R diagram is a nearly vertical straight line.**

The faint subdwarfs, Wolf 131 and Wolf 1037 discovered by Kuiper, are of spectral type K3 and K5, respectively. Their absolute magnitudes are + 12.6 and + 12.5. According to wares' estimation [6] the partially degenerate standerd model is applicable to these stars if we adopt the following as the most probable values :

$$\mathfrak{M} = 0.4\,,\ \psi_0 = +1.7 \text{ and } \mu_e = 2\,.$$

From their observed spectral types, we adopt the value $T_e^* = 0.42$. Hence we can, by using

**Strictly speaking, T_e = constant means that the denser the star is, the later its spectral type.

our table (B), obtain the following *lifetime* for Wolf 134 or Wolf 1037,

$$t_\infty - t_2 = 4.5\times10^9 \text{ years},$$

This result suggests that in about 4.5×10^9 years, the same order as the present galactic time-scale, such a subdwarf will settle down into a completely degenerate state, which will be different from the present white dwarf state, for the latter is more massive than the fomer and the latter's spectral type earlier than the former's.

According to Humason's investigation sixteen old novae, their mean spectrum is at type 0 and their mean absolute visual magnitude is about $+3.0$. According to Kuiper's theory, these stars have low hydrogen content ($\mu_e=2$), so there can be left no more effective nuclear energy source in their interior. According to wares' estimation [6] these stars have the following values :

$$\mathfrak{M} = 1.6 \text{ and } \psi_0 = -1 .$$

If we apply our calculation to these stars, by adopting $\overline{T}_c^* = 2.5$, we obtain

$$t_\infty - t_0 = 2.5\times10^8 \text{ years.} \quad \cdots\cdots (10)$$

This shows that it will take at least 2.5×10^8 years for such an old nova to evolve into a completely degenerate state. It should be noted that the derivations given for the lifetime of the old nova, Eq. (10), are based on the assumption that the star constantly keeps, throughout the whole course of its life, the present effective temperature higher than that of a usual white dwarf, so if it will take a course of lower luminosity than the present, the lifetime has to become much longer than 2.5×10^8 years given in Eq. (10).

We have no raliable data for the masses of white dwarfs at present, except Sirius B and 40 Eridani B, whose masses are about $0.98\ M_\odot$ and $0.28\ M_\odot$ respectively. If Sirius B is an "Eddington white dwarf" which consists of pure hydrogen ($\mu_e = 1$), we may take the following value : $\mathfrak{M} = 1.0$. If it can be assumed that Sirius B (spectral type A5) has kept the present effective temperature $T_c^* = 0.85$ and has evolved into the present completely degenerate state, our calculation gives the following result :

$$t_\infty - t_0 = 8.9\times10^9 \text{ years}$$

This lifetime is not shorter than the lifetime of galaxy. Therefore this result shows that the path of Sirius B in the H-R diagram lies to the far left side of the vertical line representing the spectral type A5.

Since many present astrophysicists consider that in the interior of most white dwarfs the hydrogen material has been entirely exhausted, it is of interest to assume the case of $\mu_e = 2$ for the usual white dwarfs. Therefore, for 40 Eridani B (spectral type Ao), we take the following values :

$$\mu_e = 2, \ \mathfrak{M} = 1.2 \text{ and } \overline{T}_c^* = 1.1.$$

and get, in the same way,

$$t_\infty - t_0 = 2.8\times10^8 \text{ years},$$

which is comparable with the galactic time-scale. This result shows that the path of 40 Eridani B lies to the left side of vertical line Ao in the H-R diagram.

It is believed, though not certain, that 40 Eridani B belongs to the class of the least mass in all the white dwarfs. Therefore for most of the white dwarfs we can assume the following inequality: $\mathfrak{M} \geqq 1.2$ and we get, from this,

$$t_\infty - t_0 \geqq 2.8\times10^9 \text{ years}.$$

Thus our final conclusion is that main part of the evolutionary path of a usual white dwarf lies to the left side of the verical line representing the spectral type of Ao in the H-R diagram. Consequently, it seems probable that most white dwarfs have come to the present state by cooling [9] after passing through some early spectral types such as Wolf-Rayet type, but the possibility of direct evolutionary paths from the main-sequence stars of later spectral type than Ao to the present white dwarfs is excluded, for the lifetime along such paths becomes too large compared with the galactic time-scale.

The auther wishes to express his thanks to Prof. Z. Hitotuyanagi for giving his very kind help and encouregement, and to Assistant Prof. A. Onuki and Mr. S. Kondô for their useful advices.

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THE EFFECT OF THE HEAT-TREATMENT ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD (II)

Shigeo MATSUMAE

(Received January 31, 1955)

[I] Introduction

We found that the logitudinal magnetostriction in weak magnetic field is greatly affected by the heat-treatment, according to the experimental results measured by the mutual-dynamic-impedance method[1].

Further, we measured the effect of the heat-treatment on the longitudinal magnetostriction in weak magnetic field by the capacity-bridge method. The experimental results are as follows.

[II] Experimental results and Discussions

Principle and method of the measurement are the same as described in the next paper. The principal parts of the apparatus are shown schematically in Fig. 1.

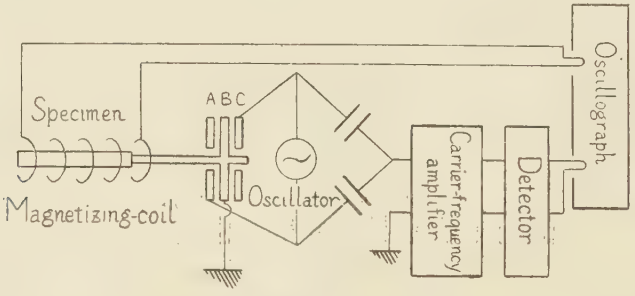


Fig. 1.

The change in length, produced by the longitudinal magnetostriction of the specimen, brings the differential-condensor plate (B) to displace and then, this displacement unbalances the capacity-bridge. The electric voltage set up by the unbalancing of the capacity-bridge is amplified by the carrier-freguency amplifier and, after detecting, recorded with the magnetizing-current on the same electromagnetic oscillograph paper. We can calculate intensities of magnetic field from one curve recorded on the oscillograph paper and magnitudes of the longitudinal magnetostriction from the other curve. From corresponding points on these two curves, we get the longitudinal magnetostrictions in many different magnetic fields by only one measurement in a short time.

The specimens are a round bars of pure nickel, 4mm in diameter and 22cm in length. The specimen are mechanically worked in the same manner and, consequently, the conditions of the internal stress in the specimens are considered to be roughly same.

The specimens are set in the electric furnace, heated from the room-temperature to 800°C in about 40 minutes by flowing the constant electric current through the heating-wire, and then hold at the annealing temperature of 800°C in the course of four different times, 5, 10, 20, 60 minutes, finally being cooled to the room-temperature with the electric-furnace by cutting the heating-current off.

As the longitudinal magnetostriction in the weak magnetic field is greatly affected by the method of demagnetizing, we take measurement of the longitudinal magnetostriction in the condition demagnetized by heating.

For example, one of the specimens which are hold at 800°C for 10 minutes shows such a longitudinal magnetostriction in Fig. 2. The similar tendency is seen in other cases.

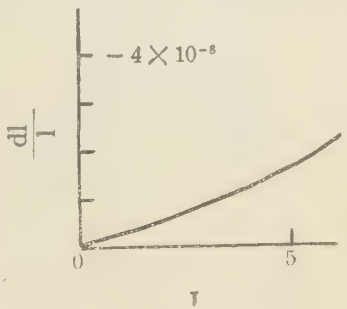


Fig. 2

When the magnitudes of the longitudinal magnetostriction for I = 4 are measured, such a result as shown in Fig. 3 is obtained. The magnitudes of the longitudinal magnetostriction in the weak magnetic field are very different even in the case of the same heat-treatment and this fact seems to be caused chiefly by the difference of the internal stress in the specimen. Consequently, the averaged value of some specimens is shown in the figure. As shown in the figure, the magnitude of the longitudinal magnetostriction in the weak magnetic field increases with the increase of

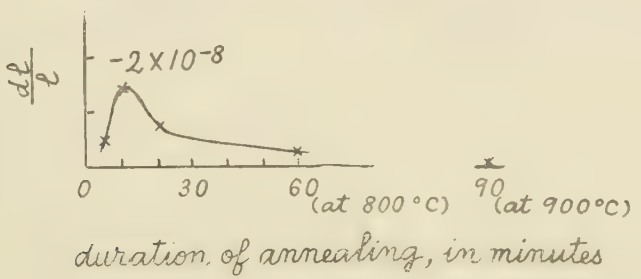


Fig. 3

the duration of annealing at first and attains a maximum value in the vicinity of 10 minutes and then decreases with the icrease of the duration of annealing. When the specimen is sufficiently annealed or, on the contrary, largly strained, the longitudinal magnetostriction in the weak magnetic field is so small as to be unobservable.

From the above, it is supposed that the longitudinal magnetostriction in the weak

magnetic field may be chiefly caused by the internal stress in the specimen.

In all cases, the longitudinal magnetostriction in the weak magnetic field decreases greatly by demagnetizing the specimen by a alternate magnetic field.

[III] Summary

According to the results of measurements, the longitudinal magnetostriction in the weak magnetic field is observed in nickel. It is greatly affected, on an average, by the heat-treatment and this fact seems to be chiefly caused by the condition of the internal stress in the specimen.

The present work is financially supported by grands in aid for the Fundamental Scientific Research of the Ministry of Education.

References.

[I] S. Matsumae: The effect of the heat-treatment on the longitudinal magnetostriction in weak magnetic field. Kumamoto Journal of Science vol. 1 no. 4. 1954.

THE EFFECT OF THE INTERNAL STRESS ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD

Shigeo MATSUMAE

(Received January 31, 1955)

(I) Introduction

We measured the longitudinal magnetostriction in the weak magnetic field in nickel and found that the logitudinal magnetostriction in the weak magnetic field seems to be chiefly affected by the internal stress in the specimen.

Consequently, we study the relation between the longitudinal magnetostriction in the weak magnetic field and the internal stress in the specimen. The experimental results are as follows.

(II) Experimental apparatus and method of measurement

(i) Principle

In the capacity-bridge, as shown in Fig. 1, $\dot{E}$ is output-voltage of a oscillator, $\dot{e}$ voltage to be measured, C_o standard condenser, $\dot{Y}_1$ and $\dot{Y}_2$ earth-admittance, C_x, C_y, C_p and C_q condensers of which the bridge is constructed.

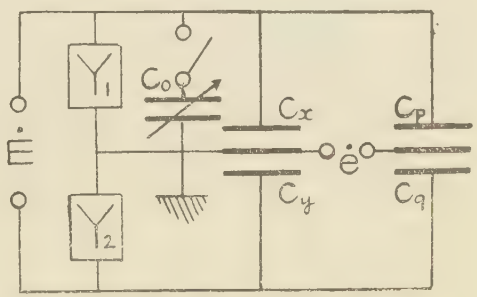


Fig. 1. Principle of the capacity-bridge for measuring the longitudinal magnetostriction

The unbalanced voltage to be measured is

$$\dot{e} = \left\{ \frac{\dot{Y}_1 + jwC_x}{\dot{Y}_1 + \dot{Y}_2 + jw(C_x + C_y)} - \frac{C_p}{C_p + C_q} \right\} \dot{E} \qquad (1)$$

When C_x and C_y change by $\triangle C_x$ and $\triangle C_y$ respectively, the above equation becomes

$$\triangle \dot{e} = \frac{\partial \dot{e}}{\partial C_x} \triangle C_x + \frac{\partial \dot{e}}{\partial C_y} \triangle C_y = \frac{jw\{(\dot{Y}_2 + jwC_y)\triangle C_x - (\dot{Y}_1 + jwC_x)\triangle C_y\}}{\{\dot{Y}_1 + \dot{Y}_2 + jw(C_x + C_y)\}^2} \dot{E} \qquad (2)$$

In the case of $\triangle C_x = C_x \frac{\triangle x}{x}$, $\triangle C_y = -C_y \frac{\triangle y}{y}$, $C_x = C_y$, $x = y$, $\triangle x = \triangle y$ and $\dot{Y}_1 = \dot{Y}_2$, we obtain

$$\triangle \dot{e} = \frac{jw C_x \dfrac{\triangle x}{x}}{2(\dot{Y}+jwC_x)} \dot{E} \qquad \cdots\cdots (3)$$

Next, when only C_x changes by $\triangle C_x$, the unbalanced voltage is

$$\triangle \dot{e}_1 = \frac{jw(\dot{Y}_2+jwC_y)\triangle C_x}{\{\dot{Y}_1+\dot{Y}_2+jw(C_x+C_y)\}^2} \dot{E}$$

Putting, as in the former case, $C_x = C_y$, $\dot{Y}_1=\dot{Y}_2$, $x=y$, we have

$$\triangle \dot{e}_1 = \frac{jw\triangle C_x}{4(\dot{Y}_1+jwC_x)} \dot{E} \qquad \cdots\cdots (4)$$

Closing the switch of the standard-condenser, making the capacitive change $\triangle C_0$ instead of $\triangle C_x$ and the unbalanced voltage $\triangle \dot{e}_0$ being equal to the above voltage $\triangle \dot{e}_1$, then we can write

$$\triangle \dot{e}_0 = \frac{jw\triangle C_0}{4(\dot{Y}_1+jwC_x)} \dot{E} \qquad \cdots\cdots (5)$$

Finally, when $\triangle \dot{e}_0 = \triangle \dot{e}$, we obtain from (3) and (5)

$$C_x \frac{\triangle x}{x} = \frac{\triangle C_0}{2}$$

$$\triangle x = \frac{x}{2} \frac{\triangle C_0}{C_x} \qquad \cdots\cdots (6)$$

If x, C_x and $\triangle C_0$ being known, we can calculate $\triangle x$ easily.

(ii) Experimental apparatus

The principal parts of the apparatus are shown schematically in Fig. 2.

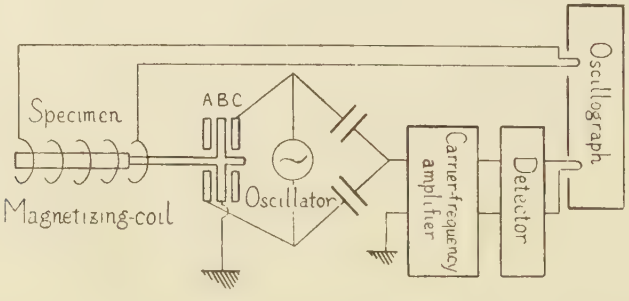


Fig. 2. Schematic diagram of the capacity-bridge method

When the change in length, produced by the longitudinal magnetostriction of the specimen, displaces the differential condenser-plate (B), this displacement unbalances the capacity-bridge. The electric voltage due to the unbalancing of the bridge is amplified by the carrier-frequency amplifier and, after detecting, is recorded with the magnetizing-current on the same electro-magnetic oscillograph paper, as shown in Fig, 6.

Each part of the apparatus is described below in detail.

(a) Magnetizing- current

The plate current of a diode is employed, in oder to get the electric current which increases continuously from zero to a certain value in a short time. By the automatic switch of the oscillograph the filament current of a diode is regulated suitably for the measurement.

(b) Oscillator

The circuit of a 555.5 KC piezo-electric oscillator is shown in Fig. 3. One-stage amplifier connectd to it is used as a buffer and also to get a high output-voltage, because the sensitivity of the bridge is proportional to the output-voltage of the oscillator, as indicated in the equation (1).

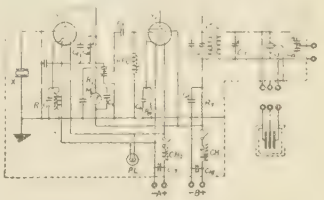


Fig. 3. Circuit of oscillator and capacity-bridge

(c) Amplifier and detector.

The circuit of amplifier and detector is shown in Fig. 4. The amplifier is designed carefully to prevent the electromagnetic wave of the oscillator from entering directly into the amplifier and to nullify, as in the case of the oscillator, effects of any external vibration and motion.

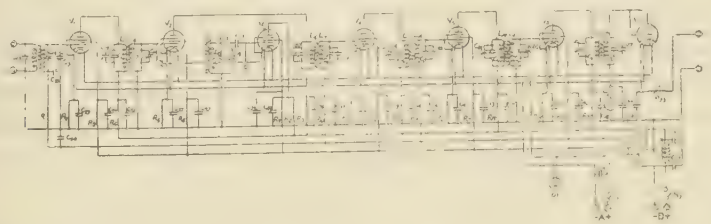


Fig. 4. Circuit of amplifier and detector.

(d) Differential condenser.

The part of the differential condenser (A) corresponds, as shown in Fig. 5, essentially to the optical-lever in the case of the optical-lever method.

(1) and (5) are disks of ebonite, 1 cm in thickness and 20 cm in diameter, in the middle of which a round hole, 2 cm in diameter, is bored and through which the differential condenser plate can be moved back and forth. (2) and (4) are round brass condenser-plates, 1.5 cm in thickness and 15 cm in diameter, in the middle of which a round hole, 2 cm in

diameter, as in the case of the ebonite-disks, is bored.

These brass condenser-plates are screwed each on these ebonite-disks which are bolted to hold the distance between these fixed condenser-plate by 6 mm. The differential condenser-plate of aluminium (3), 4 mm in thickness and 15 cm in diameter, lies between these two fixed condenser-plates and is connected to specimen by the cylindrical rod of duralumin (9),

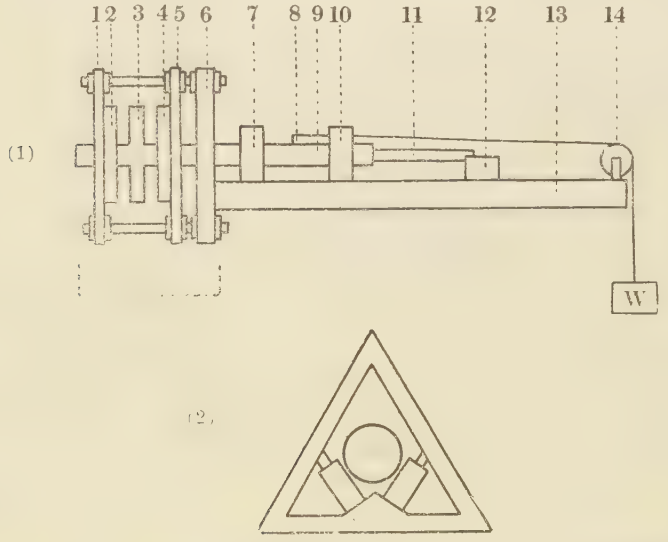


Fig. 5. (1), Experimental arrangement of sample and differential condenser. (2), portion of rollers enlarged, (7) and (10).

1.5 cm in diameter and 48 cm in length. The distances between the differential condenser-plate (3) and two fixed condenser-plates, (2) and (4), are adjusted to be each 1mm in the measurement. One end of the rod is fixed to the differential condenser and the other end is made suitable for supporting the specimen.

(e) Roller, (7) and (10)

The part of the roller is shown in the lower part of the Fig. 5. The cylindrical rod of duralumin is supported by two rollers, as seen in the figure. The rollers and its appendage are made of brass.

(f) Specimen-holder (12)

The specimen-holder is of brass and one end of the holder is made suitable for supporting the specimen.

(g) Specimen (11)

The specimen is a rod, 19 cm in length and 2.5 mm in diameter.

(h) Hook (8)

The hook is of brass and attached to the cord for applying the stress to the specimen.

(i) Frame-work (13)

The frame is a brass rod, about 1.7 m in length and 5 cm in diameter, section of which is half-round. The brass disk (6) is screwed on ond end of the frame in order to hold the part (A).

This frame is supported on the T-form block which is constructed of two wooden blocks, 1.5 m long by 15 cm square. Further, this wooden block is placed on three rubber sheets, 5 cm in thickness, to nullify any mechanical vibration and mortion.

(j) Magnetizing-coil

It is 8 cm in diameter, 1 m in length and the coil constant 116.

(k) Standard-condenser

As the standard-capacity, we use the capacitive-change which is produced by the variable condenser of a small capacity. The capacity of the condenser itself being very small compared with other capacities, it can be considered as omitted, as the consequence of the compensation of the earth-admittances ; the switch of the standard-condenser in the Fig. 1 is practically omitted and only the capacitive change of the condenser is taken into consideration as standard.

(l) Pulley (14)

The pulley is of brass.

The following matters are taken into consideration.

To be free from effects of body-capacity and other stray capacities, these apparatus are covered with earthed wire-screen or brass plates and shielded wire are employed. This is, at the same time, useful for shielding external electro-magnetic waves.

To be free from external mechanical vibration and motion, we paid attention to the amplifier and osillator, as above mentioned, and further, these apparatus are placed on rubber sheets. Electro-magnetic oscillograph and its appendages are also put on thick rubber sheets.

To be free from effects of temperature-variation, we are careful in keeping the thermal equilibrium of the room. Especially, this measurement is done in about two second.

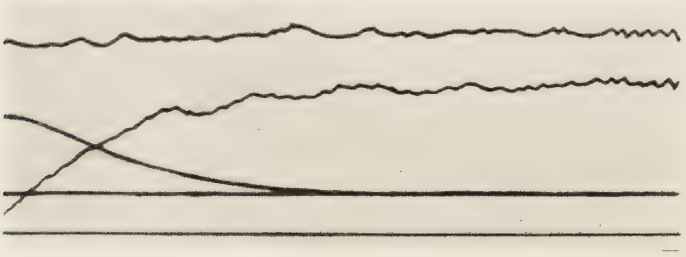
To be free from the stray magnetic field, the specimen is set apart from amplifier, oscillator and oscillograph, and these apparatus are carefully arranged.

(iii) Method of measurement.

In Fig. 2, two capacities, C_x and C_y, being adjusted alike and the unbalanced condition of the capacity and then the phase being compensated, we can get the balance of the capacity-bridge but, as the rectified current is not small even in the balanced condition, we use the D-type vibrator by applying the moderate bias voltage.

Next, we unbalance the bridge in such a degree that the unbalanced voltage due to the unbalancing is proportional to the capacitive change of the standard-condenser and then the measurement is carried out in this condition.

As shown in Fig. 6, the deflection of a vibrator produced by the magnetizing-current



Fig, 6 Oscillograph-record of the longitudinal magnetostriction in weak magnetic field.

and the deflection of another vibrator produced by the unbalanced voltage due to the displacement of the differential condenser-plate are recorded simultaneously on the same oscillograph-paper. We can calculate the intensities of the magnetizing fields readily from a curve and the magnitudes of the longitudinal magnetostrictions easily from another curve. From corresponding points on these two curves we get the longitudinal magnetostrictions in many different magnetic fields by only one measurement.

(iv) Reflections upon measurement.

The experimental result measured by the optical-lever method is compared with the result measured by the capacity-bridge method in strong magnetic field. As seen from the Fig. 7, they are almost same.

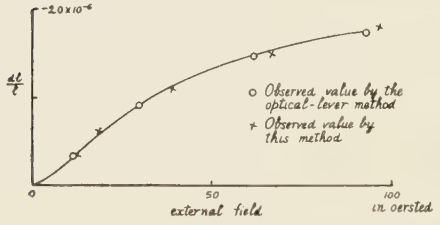


Fig. 7. Comparison of the capacity-bridge method with the optical-lever method in strong magnetic field.

But, as the longitudinal magnetostriction in weak magnetic field is very small and, consequently, the sensitivity of the measuring apparatus is very high, the matter is different in the case of the weak magnetic field.

The heat-quantity, which is generated in the specimen by the eddy-current set up during the change in magnetization, is calculated from the following equation.

$$\delta Q = (4\pi\delta I)^2 \times \frac{\lambda \times R^2}{8 \times t}$$

In this equation, δQ is the heat-quantity, δI the change of the intensity of magnetization, λ the specific electric conductivity, t the time which is spent in magnetization, R the radius of the specimen.

In this measurement, as t is about 0.5 and δI is small, the heat-quantity generated in the specimen is so little that the thermal expansion of the specimen by magnetizing need not be considered.

From the fact that, when the differential condenser-plate is fixed, the movement of the light-spot reflected by a vibrator is small, the movement of the ligth-spot in absence of the applied magnetic field is considered to be chiefly caused by the thermal expansion or contraction of the specimen and experimental arrangement.

As the measurement is carried out in about two second and ,besides, specimen and experimental arrangements have some heat-capacity, the movement of the light-spot shows an almost constant speed so that the trace of the light-spot on the oscillograph-paper is almost linear in spite of small fluctuations, as shown in Fig. 6.

It is confirmed by experiments that there is no effect of the non-uniformity of the magnetig field.

As shown in Fig. 6, fluctuations of the light-spot caused probably by noises and by external mechanical vibration and mortion are found in the traces of the light-spot on the oscillograph-paper. The results of the measurement is restricted quantitatively and qualitively by these fluctuations.

As it is almost impossible to read accurately and simultaneously both the magnitude of the capacitive change of the standard-condenser and the magnitude of the displacement of the light-spot produced by the capacitive change and, in consequence of the high sensitivity, the condition of the balancing of the capacity-bridge is continuously changed by the effect of the external temperature variation, the quantitative error is to some extent unavoidable

[III] Experimental results and Discussions

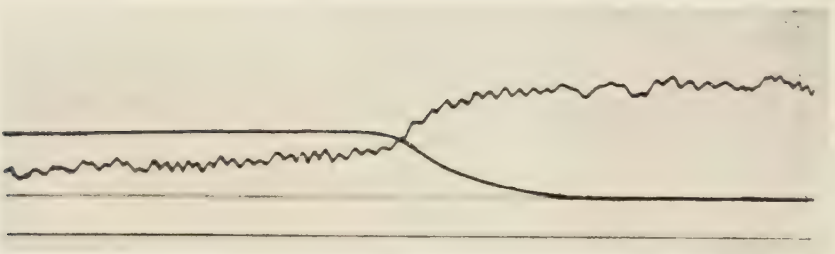
The specimen is a round bar of nickel, containning a small amount of manganese, 2.5 mm in diameter and 19 cm long.
The specimen is annealed at 900°C for 90 minutes and then cooled with the furnace to the room temperature.

As the longitudinal magnetostriction of nickel is contractive in the direction of the magnetic field, a compressive stress is applied to the specimen. In practice, the specimen is inserted in the brass cylinder, of which diameter is almost equal to that of the specimen, in order to prevent the specimen from bending and then a compressive stress is applied to the specimen.

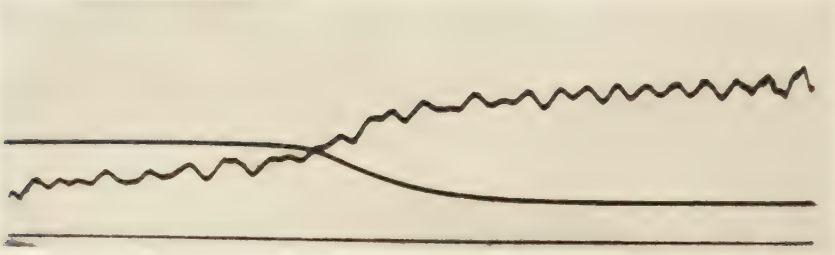
The longitudinal magnetostriction in the weak magnetic field is greatly affected by the magnitude of the maximum amplitude of a demagnetizing alternate field, if demagnetized by a alternate magnetic field. Consequently, when we take measurement of the longitudinal magnetostriction under a compressive stress, the specimen is annealed, demagnetized at first by the alertnate magnetic field having the maximum amplitude of 0.7 Oersted, subjected to a compressive stress, and then measured

The experimental results are shown in Fig. 8 and Fig. 9.

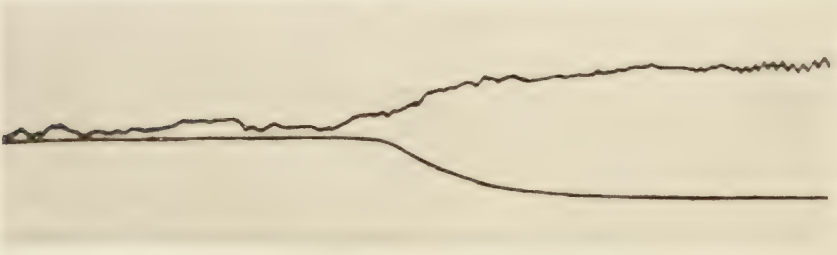
$\sigma = 0.2 \mathrm{kg/mm^2}$



$\sigma = 0.6 \mathrm{kg/mm^2}$



$\sigma = 1.2 \mathrm{kg/mm^2}$



$\sigma = 1.8 \mathrm{kg/mm^2}$

$\sigma = 2.9 \mathrm{kg/mm^2}$

Fig. 8 Oscillograph-records of the longitudinal magnetostriction in weak magnetic field under various compressive stresses.

Fig. 8 shows the records on the oscillograph-papers which are taken when various compressive stresses, 0.2kg/mm², 0.6kg/mm², 1.2kg/mm², 1.8kg/mm², 2.9kg/mm², and 3.7kg/mm², are applied individually to the specimens which are annealed at 900°C for about 90 minutes.

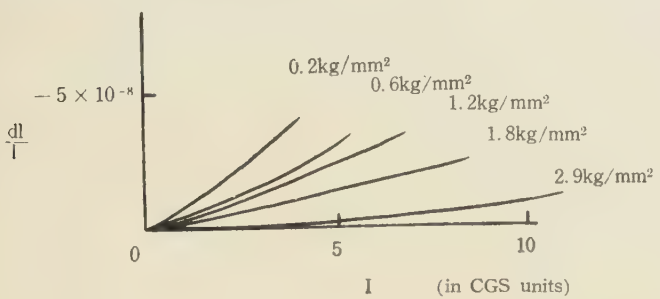


Fig. 9. $\frac{dl}{l}$ – I curves of nickel under various compressive stresses.

When the specimen is not subjected to a compressive stress, the longitudinal magnetostriction in the weak magnetic field is so small as to be unobservable but, when the specimen is subjected to a small compressive stress, the longitudinal magnetostriction in the weak magnetic field is observed, as shown in the figures. In the figures, the saturation-value of the magnetizing current is about 30 mA and, consequently, the maximum magnitude of the external magnetic field is about 0.3 Oersted. Under a compressive stress of about 200 gr/mm, the longitudinal magnetostriction in the weak magnetic field shows a maximum magnitude and then decreases with the increase of a compressive stress. Finally, under a compressive stress of 3.7 kg/mm², the logitudinal magnetostriction in the weak magnetic field is so small as to be unobservable.

Fig. 9 shows the relation between the magnitude of longitudinal magnetostriction and the intensity of magnetization in weak magnetic field under various compressive stresses, in nickel.

At first, it becomes to be clear that the longitudinal magnetostrictin in the weak magnetic field is caused by a small compressive stress. This seems to be explained in the following manner.

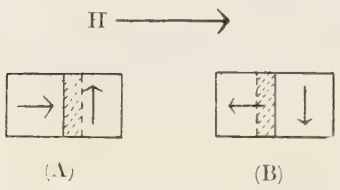


Fig. 10. Domain interpretation of the effect of a small compressive Stress on the longitudinal magnetostriction in weak magnetic field.

Fig. 10 is a highly schematic representation of the magnetic boundaries. When a weak magnetic field in the direction of the arrow is applied, the boundary displacements occur; in the case of (A), the magnetic domain oriented parallel to the direction of the magnetic field is enlarged by the displacement of the domain boundary, at the expense of the magnetic domain oriented at the right angle, and, on the other hand, in the case of (B), the reverse is the case. Consequently, the longitudinal magnetostriction does not take place in the weak magnetic field. But, when a weak magnetic field is applied to the specimen which is subjected to a small compressive stress in the direction of the magnetizing field, such a wall-movement as indicated in (A) is easier than that in (B), as nickel is magnetostrictively negative. Consequently, the longitudinal magnetostriction takes place in the weak magnetic field under a compressive stress.

Secondarily, from the fact that the longitudinal magnetostriction in the weak magnetic field decreases with the increase of a compressive stress and that the longitudinal magnetostriction in the weak magnetic field decreases greatly by demagnetizing the specimen by a alternate magnetic field of a small magnitude, it may be considered that the somewhat unstable magnetic domains oriented at about right angle to the magnetizing field seem to diminish by applying a compressive stress to the specimen, as nickel is magnetostrictively negative.

[IIII] Summary

We measure the effect of a compressive stress on the longitudinal magnetostriction in the weak magnetic field, in nickel. According to the experimental results, the longitudinal magnetostriction in the weak magnetic field is caused chiefly by a small internal stress in the specimen, in nickel.

The present work is financially supported by grands in aid for Fundamental Scientific Research of the Ministry of Education.

DIFFERENTIAL EQUATIONS IN COORDINATED SPACES.

Mituo INABA

(Received November 30, 1955)

1. *Preliminary.* We consider a sequence space E, which is an (abstract) linear space and consists of infinite sequences $x=\{x_1, x_2, \cdots\cdots, x_n, \cdots\cdots\}$, called points of the space E, where each x_n $(n=1,2,\cdots\cdots)$ is an element of each linear convex space E_n and is called the n-th coordinate of the point x. A sequence space E is called a coordinated space if it is topologized locally convex with a fundamental system of neighbourhoods $\mathfrak{U}=\{U\}$, such that each mapping $x_n(x)$: $x\to x_n$ $(n=1,2,\cdots\cdots)$ is linear and continuous. For any point $x=\{x_1, x_2, \cdots, x_n, \cdots\cdots\}$, the point, which is constructed by equating some of the coordinates of x to zero, is called a projection of the point x, and especially that which is constructed by equating the coordinates with indices greater than n to zero, is denoted by $x^{[n]}$.

We call a coordinated space E to have the property (P), if, for every neighbourhood of the origin U (of the fundamental system $\mathfrak{U}$), the relation $x\in U$ implies that [projection of x] $\in U$. For the future we shall assume this property; indeed, a great number of usual abstract spaces have it, as most Banach spaces (e.g. space (l^p) $(p\geq 1)$, space (c), space (m), and space (c), the product space of an enumerable number of spaces of real numbers, and its subspaces. For an point x of E there is considered a sequence of points $\{x^{[n]}\}$. The space E is called to have the property "Abschnittskonvergenz" or simply (AK) [1] if, for every point x, the corresponding sequence $\{x^{[n]}\}$ converges to x, that is, if, for an arbitrary neighbourhood of the origin $U\in\mathfrak{U}$, there exists a positive integer N such that $x-x^{[n]}\in U$ for $n\geq N$. Among the spaces above indicated, the first two and the product space have this property, but the second two have not. In the space E provided with the property (P), the property (AK) is stated simplified as follows: for an arbitrary neighbourhood of the origin $U\in\mathfrak{U}$, there exists a positive integer N such that $x-x^{[N]}\in U$. For the future we shall consider only coordinated spaces with the two properties (P) and (AK).

Here we introduce a lemma on (AK) for continuous functions. Let I be a compact interval $[\tau, T]$, and $x(t)$ a continuous function on I to E. Then the n-th coordinate $x_n(t)=x_n(x(t))$ is also continuous as continuous function of a continuous function, and therefore so is $x^{[n]}(t)$ as finite sum of continuous functions.

Lemma 1. *Let $x(t)$ be a continuous function on I to E. Then, $x^{[n]}(t)$ converges to $x(t)$ uniformly for $t\in I$, that is, given an arbitrary neighbourhood $U\in\mathfrak{U}$, there exists a positive integer N_0 (dependent on U, but independent of t) such that $x(t)-x^{[n]}(t)\in U$ for $n\geq N_0$.*

(1) Numbers in brackets refer to the bibliography at the end of the paper.

PROOF. For given $U \varepsilon \mathfrak{U}$, there exists a positive integer $N(U, t)$ (depending on U and t), such that for $n \geqq N$

$$x(t) - x^{[n]}(t) \varepsilon \frac{1}{2} U \quad \cdots\cdots (1)$$

Because of the continuity of the function $x(t) - x^{[N]}(t)$, there exists a positive number δ (depending on U, t and N, hence on U and t) such that the inequality $|t - t'| < \delta$ implies the relation

$$\{x(t') - x^{[N]}(t')\} - \{x(t) - x^{[N]}(t) \varepsilon \frac{1}{2} U \quad \cdots\cdots (2)$$

The relations (1) and (2) imply that

$$x(t') - x^{[N]}(t') \varepsilon U \quad \cdots\cdots (3)$$

for $|t - t'| < \delta$. In other words, for every point t of I, there corresponds a pair of a positive integer N and an open interval $I_t = \{t'; |t - t'| < \delta\}$ (a half open interval, at the terminals τ and T), such that the relations (3) holds there. By virtue of Heine-Borel covering theorem, there exist a finite set of such intervals $I_{t_1}, I_{t_2}, \cdots\cdots, I_{t_m}$ as cover the whole interval I, and a finite set of corresponding postive intergers, namely $N_1, N_2, \cdots\cdots, N_m$. Let N_0 be the maximum of these positive integers. Then, for an an arbitrary t of I, there exixts an interval I_{t_i} of the set such that $t \varepsilon I_{t_i}$, and therefore

$$x(t) - x^{[N_i]}(t) \varepsilon U,$$

by the property (P), we have

$$x(t) - x^{[N_0]}(t) \varepsilon U$$

which proves the lemma.

2. *Continuous Dependency and Uniqueness.* In this paragraph, we shall assume only the completeness of the linear convex space E. Let I be a compact interval $[o, T]$ and $f(t, x)$ a function on $I \times E$ to E. For the future we shall assume the continuity of $f(t, x)$ with respect to (t, x). Lipschitz condition shall be stated in a more general form as follows: for the function $f(t, x)$ there exists a positive, (Riemann-) integrable function $\mu(t)$, such that $\int_0^T \mu(t)\, dt < \infty$ and that, for an arbitrary neighbourhood of the origin $U \varepsilon \mathfrak{U}$, the relation $x - x' \varepsilon U$ implies that $f(t, x) - f(t, x') \varepsilon \mu(t).U$ for every pair of points x, x' belonging to the space E (or, if necessary, to a given domain of E).

The problem of an ordinary differential equation:

$$\frac{dx}{dt} = f(t, x) \quad \cdots\cdots (I)$$

under the initial conditions: $x = a$ for $t = o$, can be, as usually, reduced to the problem of the

integral equation:

$$x(t) = a + \int_0^t f(t, x(t))\, dt \quad \cdots\cdots \text{(II)}$$

In the present paper we shall assume the existence of a solution $x(t)$, instead of proving it, which may be difficult or, it seems, impossible under the conditions postulated in the present paper. The following theorem, although it does not immediately refer to the main subject of the paper, will afford the essential of the procedure of the proofs.

Theorem 1. [*Continuous dependency*] *Let* $f(t, x)$ *and* $g(t, x)$ *be functions on* $I \times E$ *to* E *and continuous, and we assume that the latter satisfies Lipschitz condition. Further, let* $x(t)$ *be a solution of the equation* (I) *and* $y(t)$ *that of the equation*

$$\frac{dx}{dt} = g(t, x) \quad \cdots\cdots \text{(I')}$$

both under the same initial condition.

Then, for a given neighbourhood of the origin $U \in \mathfrak{U}$, *the relation*

$$f(t, x) - g(t, x) \in U_0 \quad \text{for} \quad t \in I,\ x \in E$$

implies that $x(t) - y(t) \in \sigma U_0$, *where* σ *is an arbitrarily given positive number such that* $\sigma > Te^{\int_0^T \mu(t)dt}$.

PROOF From the definition of $x(t)$ and $y(t)$, it follows that

$$\begin{aligned} x(t) - y(t) &= \int_0^t [f(t, x(t)) - g(t, y(t))]\, dt \\ &= \int_0^t [f(t, x(t)) - g(t, x(t))]\, dt \\ &\quad + \int_0^t [g(t, x(t)) - g(t, y(t))]\, dt \quad \cdots\cdots (5) \end{aligned}$$

From one of the assumptions, we have

$$f(t, x(t)) - g(t, x(t)) \in U_0, \quad \cdots\cdots (6)$$

therefore

$$\int_0^t [f(t, x(t)) - g(t, x(t))]\, dt \in t U_0$$

The functions $x(t)$ and $y(t)$ are, as solutions of the differential equations, continuous functions of t, and so is the difference $x(t) - y(t)$. Since the set of points $\{x(t) - y(t);\ t \in I\}$ is compact as continuous image of the compact set I, there exists a positive number λ such that

$$x(t) - y(t) \in \lambda U_0 \quad \text{for} \quad t \in I \quad \cdots\cdots (7)$$

Lipschitz condition upon the function $g(t, x)$ and the relation (7) give the relation

$$g(t, x(t)) - g(t, y(t)) \varepsilon \lambda\mu(t) \cdot U_0, \quad \cdots\cdots (8)$$

therefore we have

$$\int_0^t [g(t, x(t)) - g(t, y(t))]\, dt \varepsilon \lambda \left(\int_0^t \mu(t)\, dt\right) \cdot U_0, \quad \cdots\cdots (9)$$

hence from the equality (5),

$$x(t) - y(t) \varepsilon (t + \lambda \int_0^t \mu(t)\, dt) \cdot U_0 \quad \cdots\cdots (10)$$

Here, if we substitute the relation (10), instead of that (7), into the left-hand side of the relation (9), we shall have, instead of the relation (9), the relation

$$\int_0^t [g(t, x(t)) - g(t, y(t))]\, dt \varepsilon \left(\int_0^t t\mu(t)dt + \lambda \int_0^t \mu(t) \int_0^t \mu(t)dt\ dt\right) \cdot U_0 \quad \cdots\cdots (9)'$$

hence, from the equality (5), we shall have the relation

$$x(t) - y(t) \varepsilon \left[t + \int_0^t t\mu(t)dt + \lambda \int_0^t \mu(t) \int_0^t \mu(t)dt\ dt\right] \cdot U_0 \quad \cdots\cdots (10)'$$

instead of the relation (10).

After p times iterrations of such procedure, we have

$$x(t) - y(t) \varepsilon \left[t + \int_0^t t\mu(t)dt + \cdots + \int_0^t \mu(t) \int_0^t \cdots \int_0^t t\mu(t)dt \cdots\cdots dt + \lambda \int_0^t \mu(t) \int_0^t \cdots\cdots \int_0^t \cdots\cdots \int_0^t \mu(t)dt \cdots\cdots dt\right] \cdot U_0 \quad \cdots\cdots (11)$$

By virture of simple calculations and an evaluation, the relation (11) can be written as follows:

$$x(t) - y(t) \varepsilon \left[t \left\{1 + \frac{1}{1!} \int_0^t \mu(t)\, dt + \frac{1}{2!} \left(\int_0^t \mu(t)\, dt\right)^2 + \cdots\cdots + \frac{1}{(p-1)!} \left(\int_0^t \mu(t)\, dt\right)^{p-1}\right\} + \frac{\lambda}{p!} \left(\int_0^t \mu(t)\, dt\right)^p\right] \cdot U \quad \cdots\cdots (12)$$

The expression enclosed with the bracket [ ] on the right-hand side of the relation (12) can be taken smaller than the beforehand given number σ, if we take the number p sufficiently large, accordingly we have

$$x(t) - y(t) \varepsilon \sigma U \quad \cdots\cdots (13)$$

which proves the theorem.

The case, in which $f(t, y) = g(t, x)$, gives the following

Corollary. [*Uniqueness.*] *Let the function $f(t, x)$ be continuous and satisfy Lipschitz condition. Then the differential equation* (I) *has at most one solution under the given*

initial condition: $x=a$ *for* $t=o$.

Remark. The relation (11) and (12) are significant in the point of view of approximations of the solution $x(t)$ of (I) by that $y(t)$ of (4). If, especially, the function $\mu(t)$ of Lipschitz condition is constant, then the situation can be more simplied, namely, the relation (11) can be replaced by the relation

$$x(t)-y(t)\varepsilon\left[\frac{1}{\mu}\left[\frac{\mu t}{1!}+\frac{(\mu t)^2}{2!}+\cdots\cdots+\frac{(\mu t)^p}{p!}+\lambda\frac{(\mu t)^p}{p!}\right]\right]\cdot U_0 \quad \cdots\cdots\cdots\cdots \quad (14)$$

Hence we have a more concrete estimation

$$x(t)-y(t)\varepsilon\sigma\cdot U_0$$

where σ is a positive function, arbitrarily taken but subject to the condition

$$\sigma \geqq \frac{e^{\mu t}-1}{\mu} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (15)$$

3. *Théorème des Réduites.* In this paragraph we assume that the space E is a coordinated space provided with the two properties (P) and (AK) and moreover that each space E_n, space of coordinate x_n, admits Picard or Peano existence theorem of the differential equation $\frac{dx_n}{dt}=f_n(t, x_n)$, as e. g. finit-dimensional spaces, Banach spaces, or spaces (M) and product spaces of spaces (M) [2].

We assume the initial condition:

$$x=a \quad \text{for} \quad t=o \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (16)$$

With a function $f(t, x)$ defined on $I\times E$, where $I=[o, T]$, we associate three functions

$$f^{[n]}(t, x),\ f(t, x^{[n]}),\ f^{[n]}(t, x^{[n]})$$

and denote each by

$$\hat{f}_{[n]}(t, x),\ \check{f}_{[n]}(t, x),\ \tilde{f}_{[n]}(x)$$

respectively. As is easily seen, the continuity of the function $f(t, x)$ implies that of the three associated functions, and the fulfillment of Lipschitz condition on the former implies that on the three associated functions with common $\mu(t)$.

Now, with a given differential equation

$$\frac{dx}{dt}=f(t, x) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (I)$$

we associate following three differential equations:

$$\frac{dx}{dt}=\hat{f}_{[n]}(t, x) \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (I_A)$$

$$\frac{dx}{dt} = \check{f}_{[n]}(t, x) \quad \cdots\cdots \text{(I.B)}$$

$$\frac{dx}{dt} = \tilde{\check{f}}_{[n]}(t, x) \quad \cdots\cdots \text{(I.C)}$$

Let each solution of (I), (I.A), (I.B) and (I.C) under the same initial condition (16) be denoted by $x(t)$, $\hat{x}_{[n]}(t)$, $\tilde{x}_{[n]}(t)$ and $\check{x}_{[n]}(t)$ respectively.

Theorem 2. *Let $f(t, x)$ be a function on $I \times E$ to E, where $I = [o, T]$, and continuous with respect to (t, x), and satisfy Lipschitz condition. Moreover we assume the existence of a solution $x(t)$ of* (I) *under the initial condition* (16).

Then the solution $x(t)$ can be approximated uniformly on the interval I by each solution $\hat{x}_{[n]}(t)$, $\tilde{x}_{[n]}(t)$, $\check{x}_{[n]}(t)$ as $n \to \infty$, i. e. for an arbitrary neighbourhood of the origin $U \in \mathfrak{U}$, there exists a positive integer N such that

$$n \geqq N \text{ implies } x(t) - \hat{x}_{[n]}(t) \in U_2 \text{ for } t \in I$$

and similarly for $\tilde{x}_{[n]}(t)$, $\check{x}_{[n]}(t)$.

PROOF (i). Let U be an arbitrary neighbourhood of the origin, and $\varkappa$ an arbitrarily chosen but fixed positive number such that

$$\varkappa < \frac{1}{T} e^{-\int_0^T \mu(t)dt}$$

where $\mu(t)$ is the function characterizing Lipschitz condition on the function $f(t, x)$. Since $x(t)$ is continuous, so is $f(t, x(t))$ and therefore by virtue of Lemma 1, there exists a positive integer N such that we have for $n \geqq N$

$$f(t, x(t)) - f^{[n]}(t, x(t)) \in U_0$$

or

$$f(t, x(t)) - \hat{f}_{[n]}(t, x(t)) \in U_0, \quad \cdots\cdots (17)$$

where $U_0 = \varkappa U$. Because of the continuity of the solutions $x(t)$ and $\hat{x}_{[n]}(t)$, there exists a positive number λ such that

$$x(t) - \hat{x}_{[n]}(t) \in \lambda U_0 \quad \cdots\cdots (18)$$

By virtue of Lipschitz condition upon $f(t, x)$ and the property (P), the relation (18) implies that

$$f^{[n]}(t, x(t)) - f^{[n]}(t, \hat{x}_{[n]}(t)) \in \lambda\mu(t) \cdot U_0$$

or

$$\hat{f}_{[n]}(t, x(t)) - \hat{f}_{[n]}(t, \hat{x}_{[n]}(t)) \in \lambda\mu(t) \cdot U_0 \quad \cdots\cdots (19)$$

Now, replacing the functions $g(t, x)$ and $y(t)$ in the proof of theorem 1 by those $f_{[n]}(t, x)$ and $\hat{x}_{[n]}(t)$ respectively, we have the relations (17), (18) and (19) corresponding to those (6), (7) and (8) respectively. Hence, according to an analogous argument there, we have

$$x(t) - \hat{x}_{[n]}(t) \in \left[t \left\{ 1 + \frac{1}{1!} \cdot \int_0^t \mu(t)\, dt + \frac{1}{2!} \left(\int_0^t \mu(t)\, dt \right)^2 \cdots\cdots \right.\right.$$

$$\left.\left. + \frac{1}{(p-1)!} \left(\int_0^t \mu(t)\, dt \right)^{p-1} \right\} + \frac{\lambda}{p!} \left(\int_0^t \mu(t)\, dt \right)^p \right] \cdot U_0 \quad \cdots\cdots\cdots \quad (20)$$

For sufficiently large p, we have

$$x(t) - \hat{x}_{[n]}(t) \in \varepsilon U \cdot U \quad \cdots\cdots\cdots \quad (21)$$

which provides the proof for $\hat{x}_{[n]}(t)$.

(ii) Again analogously to the proof of Theorem 1, we can proceed as follows. From the definition

$$x(t) - \hat{x}_{[n]}(t) = \int_0^t \left[f(t, x(t)) - \tilde{f}_{[n]}(t, x(t)) \right] dt$$

$$+ \int_0^t \left[\tilde{f}_{[n]}(t, x(t)) - \tilde{f}_{[n]}(t, \tilde{x}_{[n]}(t)) \right] dt \quad \cdots\cdots\cdots \quad (22)$$

Let U be an arbitrary neighbourhood of the origin, and ε' an arbitrarily chosen but fixed positive number such that

$$\varepsilon' < e^{-\int_0^T \mu(t) dt}$$

By virute of the continuity of the solution $x(t)$ and Lemma 1, there exists a positive integer N such that we have, for $n \geq N$,

$$x(t) - x_{[n]}(t) \in \varepsilon' U \quad \cdots\cdots\cdots \quad (23)$$

where $\varepsilon' U_0 = U$. Since $f_{[n]}(t, x(t)) = f(t, x_{[n]}(t))$, by virtue of Lipschitz condition on the function $f(t, x)$, the function (23) implies that

$$f(t, x(t)) - \tilde{f}_{[n]}(t, x(t)) \in \mu(t) \cdot U_0 \cdot \quad \cdots\cdots\cdots \quad (24)$$

Therefore

$$\int_0^t \left[f(t, x(t)) - \tilde{f}_{[n]}(t, x(t)) \right] dt \in \left(\int_0^t \mu(t)\, dt \right) \cdot U_0 \cdot \quad \cdots\cdots\cdots \quad (25)$$

Because of the continuity of $x(t) - \tilde{x}_{[n]}(t)$, as frequently observed, there exists a positive integer λ such that

$$x(t) - \tilde{x}_{[n]}(t) \in \lambda U_0, \quad \cdots\cdots\cdots \quad (26)$$

accordingly

$$x^{[n]}(t) - \tilde{x}^{[n]}_{[n]}(t) \in \lambda U_0 .$$

This relation implies, since $\tilde{f}_{[n]}(t, x) = f(t, x^{[n]})$ again, that

$$\tilde{f}_{[n]}(t, x(t)) - \tilde{f}_{[n]}(t, \tilde{x}_{[n]}) \in \lambda\mu(t)\cdot U_0 \quad \cdots\cdots (27)$$

Therefore

$$\int_0^t \left[\tilde{f}_{[n]}(t, x(t)) - \tilde{f}_{[n]}(t, \tilde{x}_{[n]}(t))\right] dt \in \lambda\left(\int_0^t \mu(t)\, dt\right)\cdot U_0 \quad \cdots\cdots (28)$$

The relations (22), (25) and (28) give the relation

$$x(t) - \tilde{x}_{[n]}(t) \in (1+\lambda)\left(\int_0^t \mu(t)\, dt\right)\cdot U_0$$

Analogously to the proof of Theorem 1, after p times iterrations, we have

$$x(t) - \tilde{x}_{[n]}(t) \in \left[\frac{1}{1!}\int_0^t \mu(t)\, dt + \frac{1}{2!}\left(\int_0^t \mu(t)\, dt\right)^2 + \cdots\cdots\right.$$

$$\left. + \frac{1}{p!}\left(\int_0^t \mu(t)\, dt\right)^p + \frac{\lambda}{p!}\left(\int_0^t \mu(t)\, dt\right)^p\right]\cdot U_0 ,$$

accordingly, since $\alpha' U_0 = U$, we have

$$x(t) - \tilde{x}_{[n]}(t) \in U.$$

which proves the second part of the theorem.

(iii) Let U be an arbitrary neighbourhood of the origin. Then, by virtue of the Lemma 1, the just proved, and the property (AK), there exists a positive integer N such that we have for $n \geqq N$

$$x(t) - \tilde{x}_{[n]}(t) \in \frac{1}{3} U, \quad \cdots\cdots (30)$$

$$x(t) - \tilde{x}_{[n]}(t) \in \frac{1}{3} U, \quad \cdots\cdots (31)$$

$$a - a^{[n]} \in \frac{1}{3} U \quad \cdots\cdots (32)$$

By virtue of the property (P) the relation (31) implies that

$$x^{[n]}(t) - \tilde{x}^{[n]}_{[n]}(t) \in \frac{1}{3} U \quad \cdots\cdots (33)$$

Since $\tilde{x}^{[n]}_{[n]}(t) = \tilde{x}^{[n]}_{[n]}(t)$ and $\tilde{x}_{[n]}(t) - \tilde{x}^{[n]}_{[n]}(t) = a - a^{[n]}$
it follows that

$$\tilde{x}_{[n]}(t) - \tilde{x}^{[n]}_{[n]} x(t) \in \frac{1}{3} U \quad \cdots\cdots (34)$$

The relations (30), (33) and (34) give the relation

$$x(t) - \tilde{\tilde{x}}_{[n]}(t) \in U \quad \cdots\cdots (35)$$

which proves the last part of the theorem. Thus the proof has been completed.

Remark. In case $a = 0$, clearly $\tilde{\tilde{x}}_{[n]}(t) = \hat{x}_{[n]} x(t)$, hence the last proof (iii) is superfluous.

Example 1. We consider a differential equation

$$\frac{dx}{dt} = x \quad \cdots\cdots (36)$$

under the initial condition (16). This satisfies the continuity and Lipschitz condition, considered as equation both in the space (m) and in the product space, and its solution is given as follows:

$$x = e^t \cdot a \quad \cdots\cdots (37)$$

In case the sequence $\{a_n\}$ of elements of $a = \{a_1, a_2, \cdots\cdots, a_n, \cdots\cdots\}$ is not convergent to zero, Theorem 2 holds in the latter space, but does not in the former, which, of course, has not the property (AK). On the contrary, in case the sequence $\{a_1, a_2, \cdots\cdots, a_n, \cdots\cdots\}$ is convergent to zero, considered as equation in the space (c_0), Theorem 2 holds, and clearly this space has the property (AK).

Example 2. We consider the differential equation (I), where $t > 0$, $|x| \leqq l$. We assume that each component $f_n(t, x)$ of the right-hand side of (I) is continuous with respect to t, uniformly for both the coordinate number n and x, and moreover satisfies the condition:

$$|f_n(t, x) - f(t, x')| \leqq \mu(t) \sum_{i=1}^{\infty} c_{ni} |x_i - x'_i| \quad \cdots\cdots (38)$$

where each series $\sum_{i=1}^{\infty} c_{ni}$ $(c_{ni} \geqq 0)$ is convergent, uniformly for the number n, and each sum is bounded by a positive constant L, and $\int_0^t \mu(t)\, dt < \infty$ for every value of T ([3], whose summary only we we know through the review of Mathematical Review.).

Considered as an equation in the space (m), the equation (I) satisfies Lipschitz condition in the form:

$$\|f(t, x) - f(t, x')\| \leqq L \mu(t) \|x - x'\|, \quad \cdots\cdots$$

but the property (AK) is not fulfilled, and therefore the result of Theorem 2 cannot be guarranteed. Now we define a norm as follows:

$$\|x\| = \sup_n \sum_{i=1}^{\infty} c_{ni} |x_i|$$

then we have $\|x\| \leqq Ll$, and Lipschitz condition in the same form as (39). The sequence space, topologized by this norm, has the property (AK). Indeed, for an arbitrary positive

number ε, there exists a positive integer N such that, by virtue of the uniformity of the convergence of the series $\sum_{i=1}^{\infty} c_{mi}$, $\sum_{i=n+1}^{\infty} c_{mi} < \frac{\varepsilon}{Ll}$ for $n \geqq N$, hence it follows

$$\| x - x^{[n]} \| = \sup_{m} \sum_{i=n+1}^{\infty} c_{mi} \cdot |x_i| \leqq \frac{\varepsilon}{l} l = \varepsilon.$$

Thus the result of Theorem 2 is ascertained.

4. *Case of General Coordinated spaces.* Let Γ be a family of indices, ι an element or index of Γ, and for each $\iota \in \Gamma$, E_ι a linear convex space. We consjder a set E whose elements x are $\{x_\iota\}_{\iota \in \Gamma}$, where $x_\iota \in E_\iota$ and is called the ι-coordinate of x. E is called a (general) coordinated space, if it is toplogized locally convex by a fundamental system of neighbourhoods $\mathfrak{U} = \{U\}$ such that each mapping $x_\iota(x)$: $x \to x_\iota$ is (i) linear, i. e. for every α, β, $x_\iota(\alpha x + \beta x') = \alpha x_\iota(x) + \beta x_\iota(x')$ and (ii) continuous, i. e. for an arbitrary neighbourhood of the origin $U \in \mathfrak{U}$ in E, there exists a neighbourhood of the origin U_ι in E_ι, such that the relation $x \in U$ implies that $x_\iota \in U_\iota$. A projection of x and the property (P) are defined analogously to the case of special coordinated spaces (Cf. 1).

Let a finite set of indices $\{\iota_1, \iota_2, \cdots\cdots, \iota_n\}$ be denoted by J, and ordering $(\leqq)$ by set-inclusion, the totality of J forms a directed system $\{J\}$, denoted by $\mathfrak{J}$. A set of points directed by $\mathfrak{J}$ is denoted by $\{x_J\}_{J \in \mathfrak{J}}$. $\mathfrak{J}$-convergence aud $\mathfrak{J}$-limit are defined as fllows: $\{x_J\}_{J \in \mathfrak{J}}$ is $\mathfrak{J}$-convegent to x (called $\mathfrak{J}$-limit of $\{x_J\}_{J \in \mathfrak{J}}$), if, for an arbitrary neighbourhood of the origin $U \in \mathfrak{U}$, there exists an element $J_0 \in \mathfrak{J}$ such that the relation $J_0 \leqq J$ implies that $x_J - x \in U$. For an element J of $\mathfrak{J}$ and a point x of E, there corresponds a projection of x, donoted by $x^{[J]}$, which is constructed dy equating all ι-coordinates $(\iota \bar{\in} J)$ of x to zero. The space E is called to have the property "$\mathfrak{J}$-Abschnittkonvergenz" or simply "Abschnittkonvergenz", if for every point to x, the corresponding directed set $\{x^{[J]}\}_{J \in \mathfrak{J}}$ is $\mathfrak{J}$-convergent to x. Specially in the coordinated space E provided with the propety (P), (AK) is stated simplied as follows: for an arbitrary neighbourhood of the origin $U \in \mathfrak{U}$, there exists an element J_0 of $\mathfrak{J}$, such that $x - x^{[J_0]} \in U$.

A lemma analogous to Lemma 1 is given, whose proof is quite the same.

Lemma 2. Let $x(t)$ be a continuous function on a compact interval I to E, which has the property (P) and that of (AK).

Then $\{x^{[J]}(t)\}_{J \in \mathfrak{J}}$ is $\mathfrak{J}$-convergent to $x(t)$ uniformly for $t \in I$, i. e. given an arbitrary neighbourhdod U, there exists an element J_0 of $\mathfrak{J}$ (dependent on U, but indepent of t) such that $x(t) - x^{[J]}(t) \in U$ for $J_0 \leqq J$.

We shall assume also in this paragraph that each space E_ι, space of coordinate x_ι, admits Picard or Peano existence theorem of the differential equation $\frac{dx_\iota}{dt} = f_\iota(t, x_\iota)$. As done in the preceding paragraph, with a function $f(t, x)$ defined on $I \times E$, associate we, for eah $J \in \mathfrak{J}$, following three functions

$$f^{[J]}(t, x),\ f(t, x^{[J]}),\ f^{[J]}(t, x^{[J]})$$

and denote each by

$$\hat{f}_{[J]}(t, x),\ \tilde{f}_{[J]}f(t, x_{[J]}),\ \check{f}_{[J]}(t, x).$$

Correspondingly, with a given differential equations

$$\frac{dx}{dt} = f(t, x) \qquad \text{(I)}$$

we can associate following three differential equations:

$$\frac{dx}{dt} = \hat{f}_{[J]}(t, x) \qquad \text{(I.}\alpha\text{)}$$

$$\frac{dx}{dt} = \tilde{f}_{[J]}(t, x) \qquad \text{(I.}\beta\text{)}$$

$$\frac{dx}{dt} = \check{f}_{[J]}(t, x) \qquad \text{(I.}\gamma\text{)}$$

and we denote each solution of (I), (I.α), (I.β) and (I.γ) under the same initial condition (16) by $x(t)$, $\hat{x}_{[J]}(t)$, $\tilde{x}_{[J]}(t)$ and $\check{x}_{[J]}(t)$ respectively.

The argument in the preceding paragraph, is, as is easily seen, quite applicable to the general coordinated space and $\mathfrak{J}$-convergence without almost any modifications. Thus we can state the following more general

Theorem 3. Let E be a complete, general coordinated space provided with two properties (P) and (AK), and I a compact interval $[0, T]$. Let $f(t, x)$ be a function on $I \times E$ to E, continuous with respect to (t, x), and satisfy Lipschitz condition. Moreover we assume the existence of a solution $x(t)$ of (I) under the initial conditions (16).

Then the solution $x(t)$ can be approximated uniformly on the interval I by each solutions $\hat{x}_{[J]}(t)$, $\tilde{x}_{[J]}(t)$, $\check{x}_{[J]}(t)$, or more precisely, for an arbitrary neighbourhood of the origin $U \in \mathfrak{U}$, there exists an element $J_0 \in \mathfrak{J}$ such that $J_0 \leqq J$ implies $x(t) - \hat{x}_{[J]}(t) \in U$ for $t \in I$ and similarly for $\tilde{x}_{[J]}(t)$ and $\check{x}_{[J]}(t)$.

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A METHOD ON THE NUMERICAL SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS

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(Received November 30, 1955)

There are many methods on the numerical solutions of a differential equation with initial conditions. But when the conditions are given with respect to several points or others, there is no systematic method and we often employ so-called "*trial and errors method*". I have established a method to find the solutions of these problems by using some approximate formulas for the differential coefficients of a function in terms of several values of the function, and few examples were shown.

1. Introduction

I have calculated some approximate formulas for differential coefficients of a function in terms of several values of the function[1], that is

$$y'_j = \sum_{i=0}^{p} B_{i,j} y_i + 0(h^p) \quad \cdots\cdots (1)$$

$$y''_j = \sum_{i=0}^{p} C_{i,j} y_i + 0(h^{p-1}) \quad \cdots\cdots (2)$$

$$j = 0, 1, 2, 3, \cdots\cdots, p$$

where $x_i = a + ih$, $y_i = f(x_i)$, $y'_i = \dfrac{d}{dx} f(x_i)$, $y''_i = \dfrac{d^2}{dx^2} f(x_i)$ and $B_{i,j}$, $C_{i,j}$ are some known constants (which were calculated). By applying these formulas for following differential equations

$$\frac{dy}{dx} = \varphi(x, y), \quad \frac{d^2y}{dx^2} = \psi\left(x, y, \frac{dy}{dx}\right)$$

with some boundary conditions, we can calculate the numerical solutions of these equations. Of course the existence of the solution must be known.

2. The Method

Consider a differential equation

$$\frac{dy}{dx} = \varphi(x, y) \quad \cdots\cdots (3)$$

under a condition

$$g(y_0, y_1, y_2, \cdots\cdots, y_m) = 0 \quad \cdots\cdots (4)$$

where $\varphi(x, y)$, $\mathcal{G}(y_0, y_1, y_2, \cdots\cdots, y_m)$ are some given functions, and let the interval of x be (a, b) and $h = (b-a)/m$, then we have following approximate equations

$$y'_{j+k} = \sum_{i=0}^{p} B_{i,j} y_{i+k}$$

where $p \leqq m$, $j = 0, 1, 2, \cdots\cdots, p$; $k = 0, 1, 2, \cdots\cdots, (m-p)$
and have following difference equations

$$\sum_{i=0}^{p} B_{i,j} y_{i+k} = \varphi(x_{j+k}, y_{j+k}) \quad \cdots\cdots (5).$$

By putting $j + k = 0, 1, 2, \cdots\cdots, m$ in (5), we get $m+1$ simultaneous equations with respect to $y_0, y_1, y_2, \cdots\cdots, y_m$.

But the determinant which made by the coefficients of y_{i+k} in the left side of (5) is zero, and the solutions of the simultaneous equations (5) are indeterminate[2]. Therefore we must take one condition in place of one of (5) and as this condition we take (4). For example if we take (4) in place of

$$\sum_{i=0}^{p} B_{i,0} y_i = \varphi(x_0, y_0)$$

then we have

$$\left.\begin{aligned} &\sum_{i=0}^{p} B_{i,j} y_{i+k} = \varphi(x_{j+k}, y_{j+k}) \\ &\qquad i + k = 1, 2, 3, \cdots\cdots, m \\ &\mathcal{G}(y_0, y_1, y_2, \cdots\cdots, y_m) = 0 \end{aligned}\right\} \quad \cdots\cdots (6).$$

(6) are $m+1$ simultaneous equations with respect to $y_0, y_1, y_2, \cdots\cdots, y_m$ and by solving these equations numerically with ordinarly method, we can obtain the values of $y_0, y_1, y_2, \cdots\cdots, y_m$.

Of course it is obvious that (5) are indeterminate with respect to $y_0, y_1, y_2, \cdots\cdots, y_m$, becouse of that the general solution of a first order differential equation has one arbitrary constant.

Similarly for a differential equation

$$\frac{d^2y}{dx^2} = \varphi\left(x, y, \frac{dy}{dx}\right) \quad \cdots\cdots (7)$$

under tow conditions

$$\left.\begin{aligned} &\mathcal{G}_1(y_0, y_1, \cdots\cdots, y_m; y'_0, y'_1, \cdots\cdots, y'_m) = 0 \\ &\mathcal{G}_2(y_0, y_1, \cdots\cdots, y_m; y'_0, y'_1, \cdots\cdots, y'_m) = 0 \end{aligned}\right\} \quad \cdots\cdots (8)$$

we have

$$\sum_{i=0}^{p} C_{i,j}\, y_{i+k} = \varphi\left(x_{j+k},\ y_{j+k},\ \sum_{i=0}^{p} B_{i,j}\, y_{i+k}\right) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (9)$$

$$j + k = 0.\ 1.\ 2.\ \cdots\cdots,\ m$$

and taking (8) and any $m-1$ equations of (9) gives $m+1$ simultaneous equations with respect to y_0, y_1, y_2, ……, y_m and by solving these equations numerically, we have the values of y_0, y_1, y_2, ……, y_m.

Consequently to find the numerical solutions of a differential equation is reduced to solve ordinary simultaneous equations numerically, and the method how to solve these simultaneous equations may be any one which is already known.

3. Errors.

Concerning the errors of the approximate formulas, we have following inequalities[3]

$$\left.\begin{aligned} |R_{p,j}| &\leqq \frac{|h^p|}{(p+1)!}\left[\sum_{i=1}^{p}\left|hB_{i,j}\right| \cdot i^{p+1} + (p+1)\, j^p\right] G \cdots\cdots \text{for } y'_j \\ |R_{p,j}| &\leqq \frac{|h^{p-1}|}{(p+1)!}\left[\sum_{i=1}^{p}\left|h^2 C_{i,j}\right| \cdot i^{p+1} + p\,(p+1)\, j^{p-1}\right] G \cdots\cdots \text{for } y''_j \end{aligned}\right\} \cdots\cdots\cdots \quad (10),$$

where G is the upper bound of $|f^{(p+1)}(x)|$ when $a \leqq x \leqq b$.

Next, the errors of the solutions of the corresponding simultaneous equations can be estimated if we do not mind the trouble of calculating, but really the procedure is very troublesome and moreover the function $y(x)$ is unknown function, hence generally it is difficult to determine the value of G accurately. But if necessary, there are some ways to assume the approximate value of G frcm the numerical silutions of $y(x)$ obtained by above method. One way is the successive application of the approximate formulas for y'_j or y''_j, that is

$$y'_s = \sum_{i=0}^{p} B_{i,j}\, y_{i+s-j}, \quad y''_s = \sum_{i=0}^{p} C_{i,j}\, y_{i+s-j}$$

$$y'''_s = \sum_{i=0}^{p} C_{i,j}\, y'_{i+s-j}, \quad y^{(iv)}_s = \sum_{i=0}^{p} C_{i,j}\, y''_{i+s-j}$$

……

……

and put the greatest value of $|y_s^{(p+1)}|$ equal to G. But as $B_{i,j}$, $C_{i,j}$ have h^{-1}, h^{-2} as their factors respectively, when we continue this method many times, their errors become very large. The second way is as follows. For example, from the equation (7), we have

$$y'''(x)=\varphi_x\big(x,y(x),y'(x)\big)+\varphi_y\big(x,y(x),y'(x)\big)y'(x)+\varphi_{y'}\big(x,y(x),y'(x)\big)y''(x)$$

and putting $x=x_s$ gives

$$y_s'''=\varphi_x(x_s,y_s,y_s')+\varphi_y(x_s,y_s,y_s')y_s'+\varphi_{y'}(x_s,y_s,y_s')y_s''.$$

By calculating the values of y', y'' from the approximate formulas and substituting them in above equation, we can calculate the value of y_s''', recurring this procedure, we can find the values of $y_s^{(iv)}$, $y_s^{(v)}$, ……, successively. Generally the latter geves very smaller errors than the former, and especially, when $\varphi(x, y, y')$ is linear with respect to y, y' and their coefficients are constants or simple forms of x, the latter is less laborious.

4. Examples.

Example. 1. Solve the equation

$$\frac{d^2y}{dx^2}=-2 \quad \cdots\cdots \text{(i)}$$

under the conditions $y(0)=y(1)=0$, $0\leqq x\leqq 1$ …… (ii).

Solution:

By the symmetry of the energy curve of (i) in $\left(y, \frac{dy}{dx}\right)$ plane with respect to y-axis, it is obvious that when $y(0)=y(1)=0$, the solution curve in (x, y) plane is symmetry with respect to the straight line $x=1/2$. Hence putting $a=0$, $b=1$, $m=10$ gives $h=1/10$, $y_{10-i}=y_i$. While when we put $p=5$, we have

$$y_0'=\frac{1}{h^2}\left(\frac{15}{4}y_0-\frac{77}{6}y_1+\frac{107}{6}y_2-13y_3+\frac{61}{12}y_4-\frac{5}{6}y_5\right)$$

$$y_1''=\frac{1}{h^2}\left(\frac{5}{6}y_0-\frac{5}{4}y_1-\frac{1}{3}y_2+\frac{7}{6}y_3-\frac{1}{2}y_4+\frac{1}{12}y_5\right)$$

$$y_2''=\frac{1}{h^2}\left(-\frac{1}{12}y_0+\frac{4}{3}y_1-\frac{5}{2}y_2+\frac{4}{3}y_3-\frac{1}{12}y_4\right)$$

$$y_3''=\frac{1}{h^2}\left(\frac{1}{12}y_1+\frac{4}{3}y_2-\frac{5}{2}y_3+\frac{4}{3}y_4-\frac{1}{12}y_5\right)$$

$$y_4''=\frac{1}{h^2}\left(-\frac{1}{12}y_2+\frac{4}{3}y_3-\frac{5}{2}y_4+\frac{4}{3}y_5-\frac{1}{12}y_6\right)$$

$$y_5''=\frac{1}{h^2}\left(-\frac{1}{12}y_3+\frac{4}{3}y_4-\frac{5}{2}y_5+\frac{4}{3}y_6-\frac{1}{12}y_7\right) \quad \cdots\cdots \text{(iii)}.$$

$$y_6''=\frac{1}{h^2}\left(-\frac{1}{12}y_4+\frac{4}{3}y_5-\frac{5}{2}y_6+\frac{4}{3}y_7-\frac{1}{12}y_8\right)$$

$$y_7''=\frac{1}{h^2}\left(-\frac{1}{12}y_5+\frac{4}{3}y_6-\frac{5}{2}y_7+\frac{4}{3}y_8-\frac{1}{12}y_9\right)$$

$$y_8''=\frac{1}{h^2}\left(-\frac{1}{12}y_6+\frac{4}{3}y_7-\frac{5}{2}y_8+\frac{4}{3}y_9-\frac{1}{12}y_{10}\right)$$

$$y_9''=\frac{1}{h^2}\left(\frac{1}{12}y_5-\frac{1}{2}y_6+\frac{7}{6}y_7-\frac{1}{3}y_8-\frac{5}{4}y_9+\frac{5}{6}y_{10}\right)$$
$$y_{10}''=\frac{1}{h^2}\left(-\frac{5}{6}y_5+\frac{61}{12}y_6-13y_7+\frac{107}{6}y_8-\frac{77}{6}y_9+\frac{15}{4}y_{10}\right)$$

Substituting $y_0=y_{10}=0$, $y_6=y_4$, $y_7=y_3$, $y_8=y_2$, $y_9=y_1$ in above equations gives following five equations

$$\left.\begin{aligned}
-\frac{5}{4}y_1-\frac{1}{3}y_2+\frac{7}{6}y_3-\frac{1}{2}y_4+\frac{1}{12}y_5&=-0.02\\
\frac{4}{3}y_1-\frac{5}{2}y_2+\frac{4}{3}y_3-\frac{1}{12}y_4&=-0.02\\
-\frac{1}{12}y_1+\frac{4}{3}y_2-\frac{5}{2}y_3+\frac{4}{3}y_4-\frac{1}{12}y_5&=-0.02\\
-\frac{1}{12}y_2+\frac{4}{3}y_3-\frac{31}{12}y_4+\frac{4}{3}y_5&=-0.02\\
-\frac{1}{6}y_3+\frac{8}{3}y_4-\frac{5}{2}y_5&=-0.02
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\quad\text{(iv)}$$

and by solving these equations, we have $y_1=0.090000$, $y_2=0.160000$, $y_3=0.210000$, $y_4=0.240000$, $y_5=0.250000$.

Of course we can solve (i) analytically and the result is $y=x(1-x)$ and above values satisfy this equation exactly. Namely from (i) we have $y^{(5+1)}(x)\equiv 0$ and $G=0$, hence $R=0$ and the equation (iv) have no error.

Example. 2. Solve the equation

$$\frac{d^2y}{dx^2}=-\sqrt{10+\left(\frac{dy}{dx}\right)^2}\quad\cdots\cdots\cdots\cdots\quad\text{(i)}$$

under the conditions

$$y(0)=y(1)=0,\quad 0\leqq x\leqq 1\quad\cdots\cdots\cdots\cdots\quad\text{(ii)}.$$

Solution:

Similarly as above example, we can obtain following equations

$$\left.\begin{aligned}
-\frac{5}{4}y_1-\frac{1}{3}y_2+\frac{7}{6}y_3-\frac{1}{2}y_4+\frac{1}{12}y_5&=Z_1\\
\frac{4}{3}y_1-\frac{5}{2}y_2+\frac{4}{3}y_3-\frac{1}{12}y_4&=Z_2\\
-\frac{1}{12}y_1+\frac{4}{3}y_2-\frac{5}{2}y_3+\frac{4}{3}y_4-\frac{1}{12}y_5&=Z_3\\
-\frac{1}{12}y_2+\frac{4}{3}y_3-\frac{31}{12}y_4+\frac{4}{3}y_5&=Z_4\\
-\frac{1}{6}y_3+\frac{8}{3}y_4-\frac{5}{2}y_5&=Z_5
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\quad\text{(iii)}$$

where $Z_i = -0.01 \times \sqrt[3]{60 + y_i'^2}$, $i = 1, 2, 3, 4, 5$
and

$$\left.\begin{aligned}
y_1' &= 10\left(-\frac{13}{12}y_1 + 2y_2 - y_3 + \frac{1}{3}y_4 - \frac{1}{20}y_5\right)\\
y_2' &= 10\left(-\frac{1}{2}y_1 - \frac{1}{3}y_2 + y_3 - \frac{1}{4}y_4 + \frac{1}{30}y_5\right)\\
y_3' &= 10\left(\frac{1}{4}y_1 - y_2 + \frac{1}{3}y_3 + \frac{1}{2}y_4 - \frac{1}{20}y_5\right)\\
y_4' &= 10\left(-\frac{1}{3}y_1 + y_2 - 2y_3 + \frac{13}{12}y_4 + \frac{1}{5}y_5\right)\\
y_5' &= 10\left(\frac{5}{4}y_1 - \frac{10}{3}y_2 + 5y_3 - 5y_4 + \frac{137}{60}y_5\right)
\end{aligned}\right\} \quad \cdots\cdots \text{(iv)}$$

To solve these simultaneous equations exactly is so difficult that we employ following iterative method.

Consider the equations

$$\left.\begin{aligned}
-\frac{5}{4}y_{1,n} - \frac{1}{3}y_{2,n} + \frac{7}{6}y_{3,n} - \frac{1}{2}y_{4,n} + \frac{1}{12}y_{5,n} &= Z_{1,n-1}\\
\frac{4}{3}y_{1,n} - \frac{5}{2}y_{2,n} + \frac{4}{3}y_{3,n} - \frac{1}{12}y_{4,n} &= Z_{2,n-1}\\
-\frac{1}{12}y_{1,n} + \frac{4}{3}y_{2,n} - \frac{5}{2}y_{3,n} + \frac{4}{3}y_{4,n} - \frac{1}{12}y_{5,n} &= Z_{3,n-1}\\
-\frac{1}{12}y_{2,n} + \frac{4}{3}y_{3,n} - \frac{31}{12}y_{4,n} + \frac{4}{3}y_{5,n} &= Z_{4,n-1}\\
-\frac{1}{6}y_{3,n} + \frac{8}{3}y_{4,n} - \frac{5}{2}y_{5,n} &= Z_{5,n-1}
\end{aligned}\right\} \quad \cdots\cdots \text{(v)}$$

and at first we give any values to $y_{1,0}, y_{2,0}, \cdots\cdots y_{5,0}$ as the first approximate values of $y_1, y_2, \cdots\cdots, y_5$ and calculate the values of $Z_{1,0}, Z_{2,0}, \cdots\cdots, Z_{5,0}$ from (iv) and substitute these values into (v), then they become five simultaneous equations of first degree with respect to $y_{1,1}, y_{2,1}, \cdots\cdots, y_{5,1}$ and by solving these equations with ordinarly method, we obtain the values of $y_{1,1}, y_{2,1}, \cdots\cdots, y_{5,1}$, recurring this procedure we have the values of $y_{1,2}, y_{2,2}, \cdots\cdots, y_{1,n}, y_{2,n}, \cdots\cdots, y_{5,n}$ successively and if $\lim_{n\to\infty} y_{i,n}$ converge to definite values y_i, then these values are the roots of (iii). Starting from the initial values $y_{1,0} = 0.1$, $y_{2,0} = 0.2$, $y_{3,0} = 0.3$, $y_{4,0} = 0.4$, $y_{5,0} = 0.5$ the result of my calculation is as follows, and they show that $y_{i,4} \equiv y_{i,5}$.

Table 1.

1	0.1	1.00	0.3509	3.118	0.36027	3.1802
2	0.2	1.00	0.6236	2.337	0.63699	2.3585
3	0.3	1.00	0.8183	1.556	0.83253	1.5596
4	0.4	1.00	0.9348	0.775	0.94941	0.7761
5	0.5	1.00	0.9734	−0.004	0.98818	−0.0002

i	$y_{i,3}$	$y'_{i,3}$	$y_{i,4}$	$y'_{i,4}$	$y_{i,5}$	Y
1	0.360596	3.18165	0.360603	3.18171	0.360603	0.360603
2	0.637389	2.35882	0.637398	2.35883	0.637398	0.637396
3	0.833145	1.55956	0.833153	1.55956	0.833153	0.833150
4	0.949820	0.77643	0.949827	0.77642	0.949827	0.949824
5	0.988582	0.00002	0.988590	−0.00001	0.988590	0.988587

The equation (i) can be solved directly also and the result is

$$y = \sqrt{60}\left\{\cosh\frac{1}{2} - \cosh\left(x - \frac{1}{2}\right)\right\}$$

and the last figures Y in the table. 1. are the values which were calculated by this formula and they show that $y_{i,5} \doteqdot Y$.

Example. 3. Find the periodic solution of the differential epuation

$$\frac{d^2y}{dx^2} = -12\sin y \qquad \text{(i)}$$

under the conditions

$$y(0) = 0, \text{ and period} = 2 \qquad \text{(ii)}.$$

Solution:

It is easily shown that (i) has a periodic solution and its period T is any value not less than $2\pi/\sqrt{12}$. When we put $a = 0$, $T = 2$, $(2\pi/\sqrt{12} < 2)$, then by the symmetry of energy curve in the phase plane, we can put $b = 2$, $y(0) = y(1) = y(2) = 0$, and $y_i = y_{10-i} = -y_{10+i}$, $(i = 0, 1, 2, \cdots\cdots, 10)$, where $m = 20$, therefore it is sufficient that we calculate y_1, y_2, y_3, y_4, y_5. Similarly as above examples, putting $p = 5$ gives

$$\left.\begin{aligned}
-\frac{5}{4}y_1 - \frac{1}{3}y_2 + \frac{7}{6}y_3 - \frac{1}{2}y_4 + \frac{1}{12}y_5 &= Z_1\\
\frac{4}{3}y_1 - \frac{5}{2}y_2 + \frac{4}{3}y_3 - \frac{1}{12}y^4 &= Z_2\\
-\frac{1}{12}y_1 + \frac{4}{3}y_2 - \frac{5}{2}y_3 + \frac{4}{3}y_4 - \frac{1}{12}y_5 &= Z_3\\
-\frac{1}{12}y_2 + \frac{4}{3}y_3 - \frac{31}{12}y_4 + \frac{4}{3}y_5 &= Z_4\\
-\frac{1}{6}y_3 + \frac{8}{3}y_4 - \frac{5}{2}y_5 &= Z_5
\end{aligned}\right\} \qquad \text{(iii)}$$

where $Z_i = -0.12\sin y_i$, $\quad i = 1, 2, 3, 4, 5.$

Table 2.

i	$y_{i,0}$	$y_{i,1}$	$y_{i,2}$	$y_{i,3}$	$y_{i,4}$	$y_{i,5}$	$y_{i,6}$	$y_{i,7}$	$y_{i,8}$
1	0.2	0.2729	0.3138	0.3325	0.3500	0.3636	0.3729	0.3792	0.3830
2	0.4	0.5220	0.5826	0.6288	0.6620	0.6872	0.7045	0.7159	0.7231
3	0.6	0.7246	0.7988	0.8597	0.9043	0.9380	0.9609	0.9761	0.9855
4	0.8	0.8597	0.9361	1.0054	1.0569	1.0953	1.1215	1.1388	1.1495
5	1.0	0.9091	0.9831	1.0549	1.1088	1.1488	1.1760	1.1939	1.2051

By similar iterative method in ex. 2., we calculated $y_{i,1}$, $y_{i,2}$, ……, $y_{i,n}$ successively, and have gotten the result which is shown in the table. 2.

As this table shows, the convergence of $y_{i,n}$ are very slow and it seems so laborious to continue this iteration that we take following method.

If we put

$$\left.\begin{aligned} &y_1 = 0.385 + \varepsilon_1,\ y_2 = 0.727 + \varepsilon_2,\ y_3 = 1.000 + \varepsilon_3 \\ &y_4 = 1.160 + \varepsilon_4,\ y_5 = 1.220 + \varepsilon_5 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \text{(iv)}$$

then ε_1, ε_2, ε_3, ε_4, ε_5 may be very small[4], and

$$\left.\begin{aligned} Z_1 &= -0.12sin(0.385+\varepsilon_1) \fallingdotseq -0.12sin\,0.385 - (0.12cos\,0.385)\varepsilon_1 \\ Z_2 &= -0.12sin(0.727+\varepsilon_2) \fallingdotseq -0.12sin\,0.727 - (0.12cos\,0.727)\varepsilon_2 \\ Z_3 &= -0.12sin(1.000+\varepsilon_3) \fallingdotseq -0.12sin\,1.000 - (0.12cos\,1.000)\varepsilon_3 \\ Z_4 &= -0.12sin(1.160+\varepsilon_4) \fallingdotseq -0.12sin\,1.160 - (0.12cos\,1.160)\varepsilon_4 \\ Z_5 &= -0.12sin(1.220+\varepsilon_5) \fallingdotseq -0.12sin\,1.220 - (0.12cos\,1.220)\varepsilon_5 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \text{(v)}$$

and substituting (iv), (v) into (iii) gives

$$\left.\begin{aligned} -1.1388\,\varepsilon_1 - 0.3333\,\varepsilon_2 + 1.1667\,\varepsilon_3 - 0.5000\,\varepsilon_4 + 0.0833\,\varepsilon_5 &= -0.009817 \\ 1.3333\,\varepsilon_1 - 2.4103\,\varepsilon_2 + 1.3333\,\varepsilon_3 - 0.0833\,\varepsilon_4 &= -0.012256 \\ -0.0833\,\varepsilon_1 + 1.3333\,\varepsilon_2 - 2.4352\,\varepsilon_3 + 1.3333\,\varepsilon_4 - 0.0833\,\varepsilon_5 &= 0.016773 \\ -0.0833\,\varepsilon_2 + 1.3333\,\varepsilon_3 - 2.5354\,\varepsilon_4 + 1.3333\,\varepsilon_5 &= -0.012766 \\ -0.1667\,\varepsilon_3 + 2.6667\,\varepsilon_4 - 2.4588\,\varepsilon_5 &= 0.010642 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \text{(vi)}$$

and we can solve these equations exactly and the result is

$$\varepsilon_1 = 0.0044,\quad \varepsilon_2 = 0.0076,\quad \varepsilon_3 = 0.0007,$$

$$\varepsilon_4 = 0.0066,\quad \varepsilon_5 = 0.0028$$

therefore

$$y_1 = 0.3894,\quad y_2 = 0,7346,\quad y_3 = 1.0007,$$

$$y_4 = 1.1666,\quad y_5 = 1.2228$$

If α is the amplitude of the solution, then its period T is expressed by

$$T = \frac{2}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - sin^2 \frac{\alpha}{2}\, sin^2\theta}} .$$

Putting $T = 2$ gives

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\theta}} = \sqrt{3} \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{(vii)}$$

and from the table of the values of elliptic integral, we obtain

$$\alpha \fallingdotseq 70^\circ 8' \fallingdotseq 1.2241$$

this is nearly epual to y_6 which must be the amplitude.

5. Comments.

It is evident that when the simultaneous equations are linear with respect to y_0, y_1, y_2, ..., y_m, we can calculate their roots exactly, and even if they are not linear, in most cases we can calculate the numerical solutions of them at any accuracy. But there are some errors in (5) or (9), and if we wish to have more precise solutions of the given differential equation, h must be small and we must take the approximate formulas of the differential coefficients for large p. It is noticeable fact that the left sides of the equations (iv) in ex. 1, (iii) in ex. 2, and (iii) in ex. 3, are entirely the same.

References

(1), (2) and (3)——R. Adachi: Approximate Formulas for Definite Integrals and Differential Coefficients, Kumamoto J of S. Vol. 2, No. 2, 1955.

(4)——Some conditions must be added in orded to that they are very small.
See—R. Adachi: On the Newton's Method for the Approximate Solutions of Simultaneous Equations in this Journal.

FUNDAMENTAL RELATIONS ON THE SEISMIC PROSPECTING

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(Received November 30, 1955)

1. Introduction.

The seismic prospecting is one method which makes us to determine the form of the surface of separation from the travelling-time curve and it is based on the fact that if the surface of separation is a plane (therefore its horizontal section is a straight line), the travelling-time curve is a straight line[1]. But inversely, when the travelling-time curve is a straight line, is the horizontal section a straight line ? I researched the relation between the curve of separation and the travelling-time curve when two layers and three layers, and have obtained general relations between these curves, and from these relations, it is easily proved that above inverse is true when two layers, and some remarks were given.

2. General Relations When Three Layers.

Let the surface of separation be two cylinders which are made by a generating line moving parallel to a fixed horizontal line l, and let

$$y = g(x) \cdots\cdots\cdots\cdots (1)$$

$$y = f(x) \cdots\cdots\cdots\cdots (2)$$

be the equations of the sections of the surfaces by a plane which is perpendicular to l. As shown in Fig. 1, let $OTPQSR$ be a path along which a wave-ray is propagated from O to R, and its travelling-time be t, refracting at P with the incident angle θ_2 and then running along the curve $y = f(x)$ with the velocity v_3 and at Q the wave refracts at the same angle, where v_1, v_2, v_3 are the velocities of the wave in the first layer the second layer and the third layer respectively, and $v_1 < v_2 < v_3$.

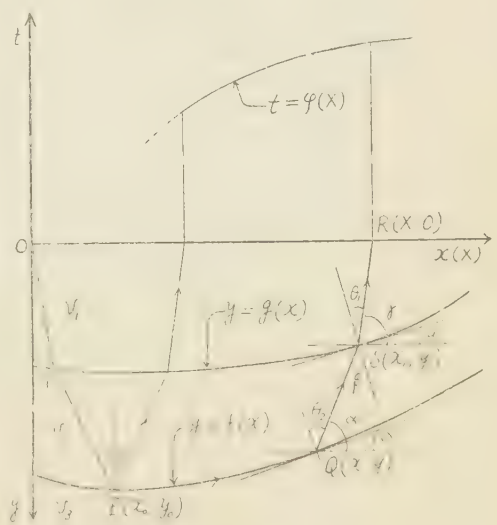

Fig. 1.

Putting α, β, γ, ω, σ, θ_1 as shown in Fig. 1, we have following relations

$$\left.\begin{aligned} &\alpha = \frac{\pi}{2} - \sigma - \theta_2, \qquad \sin\theta_2 = \frac{v_2}{v_3}, \qquad \sin\theta_1 = \frac{v_1}{v_2}\sin\beta \\ &\beta = \frac{\pi}{2} - \omega - \alpha, \\ &\gamma = \frac{\pi}{2} - \omega - \theta_1, \qquad \tan\omega = \frac{d}{dx}g(x_1), \qquad \tan\sigma = \frac{d}{dx}f(x) \end{aligned}\right\} \cdots\cdots (3).$$

Let $m = -\tan\alpha$, then we have

$$m = -\tan\alpha = \frac{v_2 f'(x) - V_2}{v_2 + V_2 f'(x)} \cdots\cdots (4)$$

$$1 + m^2 = \frac{v_3^2\,[1 + f'(x)^2]}{[v_2 + V_2 f(x)]^2} \cdots\cdots (4)',$$

where $V_2 = \sqrt{v_3^2 - v_2^2}$. Now, x_1, y_1 are the solutions of followion simultaneous equations

$$\left.\begin{aligned} g(x_1) - f(x) &= m(x_1 - x) \\ y_1 &= g(x_1) \end{aligned}\right\} \cdots\cdots (5)$$

and we have

$$\overline{QS} = \sqrt{1 + m^2}\,(x_1 - x), \qquad \overline{SR} = y_1/\sin\gamma \cdots\cdots (6)$$

$$X = x_1 + y_1\cot\gamma \cdots\cdots (7)$$

$$t = t_0 + \frac{1}{v_3}\int_{x_0}^{x}\sqrt{1 + f'^2(x)}\,dx + \frac{1}{v_2}\overline{QS} + \frac{1}{v_1}\overline{SR} \cdots\cdots (8)$$

and let the travelling-time curve be

$$t = \varphi(X) \cdots\cdots (9).$$

3. When $f(x)$ and $g(x)$ are given.

By eliminating x, y, x_1, y_1, α, β, γ, σ, ω, θ_1 and m from (3), (4), (5), (6), (7), and (8) we get a relation between t and X which is the travelling-time curve (9).

4. When $\varphi(X)$ is given.

From (8) and (9) we have

$$t_0 + \frac{1}{v_3}\int_{x_0}^{x}\sqrt{1 + f'(x)^2}\,dx + \frac{1}{v_2}\overline{QS} + \frac{1}{v_1}\overline{SR} = \varphi(X) \cdots\cdots (10)$$

and differentiating (10) with respect to x gives

$$\frac{1}{v_3}\sqrt{1+f'(x)^2}+\frac{1}{v_2}\frac{d}{dx}\overline{QS}+\frac{1}{v_1}\frac{d}{dx}\overline{SR}=\varphi'(X)\frac{dX}{dx} \quad \cdots\cdots\cdots\cdots (11).$$

While

$$\left.\begin{aligned}
&\frac{dm}{dx}=\frac{v_3^2}{[v_2+V_2f'(x)]^2}f''(x)\\
&\frac{dx_1}{dx}=\frac{1}{\tan\omega-m}\cdot\left[f'(x)-m+(x_1-x)\frac{dm}{dx}\right]\\
&\frac{dy_1}{dx}=\frac{dx_1}{dx}\tan\omega=\frac{1}{\tan\omega-m}\cdot\left[f'(x)-m+(x_1-x)\frac{dm}{dx}\right]\tan\omega\\
&\frac{d\beta}{dx}=-\frac{d\alpha}{dx}=\frac{1}{1+f'(x)^2}\cdot f''(x)\\
&\frac{d\theta_1}{dx}=\frac{v_1\cos\beta}{v_2\cos\theta_1}\cdot\frac{f''(x)}{1+f'(x)^2},\quad \frac{d\gamma}{dx}=-\frac{d\theta_1}{dx}=-\frac{v_1\cos\beta}{v_2\cos\theta_1}\cdot\frac{f''(x)}{1+f'(x)^2}
\end{aligned}\right\} \quad \cdots\cdots (12)$$

and

$$\left.\begin{aligned}
&\frac{dX}{dx}=\frac{dx_1}{dx}+\frac{dy_1}{dx}\cot\gamma-\frac{1}{\sin^2\gamma}y_1\frac{d\gamma}{dx}\\
&\frac{d}{dx}\overline{QS}=\frac{1}{\sqrt{1+m^2}}\left[(x_1-x)m\frac{dm}{dx}+(1+m^2)\left(\frac{dx_1}{dx}-1\right)\right]\\
&\frac{d}{dx}\overline{SR}=\frac{1}{\sin\gamma}\frac{dy_1}{dx}-\frac{\cos\gamma}{\sin^2\gamma}y_1\frac{d\gamma}{dx}
\end{aligned}\right\} \quad \cdots\cdots (13)$$

Applying these relations in (11) and troublesome arrangement give

$$\left[\frac{\cos\gamma}{v_1}-\varphi'(X)\right]\frac{dX}{dx}=0$$

therefore

$$\frac{\cos\gamma}{v_1}-\varphi'(X)=0 \qquad or \quad \frac{dX}{dx}=0.$$

(i) When $\dfrac{\cos\gamma}{v_1}-\varphi'(X)=0.$

In this case, we get

$$\sqrt{v_2^2(1+m^2)-v_1^2(\cos\omega+m\sin\omega)^2}\cdot\sin\omega+v_1(\cos\omega+m\sin\omega)\cos\omega$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (14)$$

where

$$m = \frac{v_2 f'(x) - V_2}{v_2 + V_2 f'(x)}, \quad \tan\omega = g'(x_1)$$

$$g(x_1) - f(x) = m(x_1 - x) \qquad (15)$$

$$X = x_1 + \frac{\sqrt{v_2^2(1+m^2) - v_1^2(\cos\omega + m\sin\omega)^2}\cdot\sin\omega + v_1(\cos\omega + m\sin\omega)\cos\omega}{\sqrt{v_2^2(1+m^2) - v_1^2(\cos\omega + m\sin\omega)^2}\cdot\cos\omega - v_1(\cos\omega + m\sin\omega)\sin\omega} g(x_1)$$

If $\varphi(X)$ is given, (14) and (15) are functional relations between $f(x)$, $g(x_1)$ and one of them is solved provided the other is given.

(a) When $g(x_1)$ is given.

From (15), x_1 and ω can be expressed by $f(x)$, $f'(x)$ and x, and (14) becomes a relation of x, $f(x)$ and $f'(x)$ and this is a differential equation of $f(x)$.

(b) When $f(x)$ is given.

From (15), x and m can be expressed by x_1 and $g(x_1)$, and (14) becomes a relation of x_1, $g(x_1)$ and $g'(x_1)$ and this is a differential equation of $g(x_1)$. When $\varphi(X)$ is a linear function of X, that is the travelling-time curve is a straight line, we have $\varphi'(X) = const$, and moreover if $g(x_1)$ is a linear function of x_1, we have $g'(x_1) = canst$, $\omega = const$ and from (14) we get $f'(x) = const$, therefore $f(x)$ is also a linear function of x. Inversely if $f(x)$ is a linear function of x, then $f'(x) = const$ and $m = const$, and from (14) we get $g'(x_1) = const$, therefore $g(x_1)$ is a linear function of x_1 also. It is obvious that if the linearity of $\varphi(X)$ is given only, none is introducet abont $f(x)$ and $g(x)$. When $f(x)$ and $g(x)$ are linear functions, from (14) and (15), we have $\varphi'(X) = const$ and $\varphi(X)$ is a linear function of X.

(ii) When $\frac{dX}{dx} = 0$.

We get $X = const = X_0$ say, and have

$$x_1 + \frac{\sqrt{v_2^2(1+m^2) - v_1^2(\cos\omega + m\sin\omega)^2}\cdot\sin\omega + v_1(\cos\omega + m\sin\omega)\cos\omega}{\sqrt{v_2^2(1+m^2) - v_1^2(\cos\omega + m\sin\omega)^2}\cdot\cos\omega - v_1(\cos\omega + m\sin\omega)\sin\omega} \cdot g(x_1) = X_0 \qquad \cdots\cdots (17)$$

$$\left.\begin{aligned} m &= \frac{v_2 f'(x) - V_2}{v_2 + V_2 f'(x)}, \quad \tan\omega = g'(x_1) \\ g(x_1) - f(x) &= m(x_1 - x) \end{aligned}\right\} \cdots\cdots (18).$$

If $g(x_1)$ is given, we can express x_1 and ω as a function of $f(x)$, $f'(x)$ and x, and substituting them in (17), we get a differential equation of $f(x)$. Of course $X = const$ means that the travelling-time curve is a straight line perpendicular to the X-axis, and this

case almost does not occur in practice.

5. When Two Layers.

If we put to $v_1 = v_2$ in the preceding article, we get the case of two layers. In the equations (14) and (15), putting to $v_1 = v_2$ we get

$$v_2 + V_2 f'(x) - v_2 v_3 \sqrt{1 + f'(x)^2}\, \varphi'\left(x + \frac{v_2 + V_2 f'(x)}{V_2 - v_2 f'(x)} f(x)\right) = 0 \quad \cdots\cdots \quad (19)$$

$$X = x + \frac{v_2 + V_2 f'(x)}{V_2 - v_2 f'(x)} f(x) \quad \cdots\cdots \quad (20)$$

and from (17) and (18)

$$f(x) = \frac{v_2 f'(x) - V}{v_2 + V_2 f'(x)} (x - X) \quad \cdots\cdots \quad (21).$$

These are the differential equations with respect to $f(x)$ when the travelling-time curve $\varphi(X)$ is given. In the equation (21), by changing (x, y) to the polar co-ordinate (r, Θ) and integrating, we get

$$r = C \exp(k\Theta) \quad \cdots\cdots \quad (22)$$

where $k = v_2/V_2$ and $C = const$, and this is an equiangular spiral. Of course previously stated, this case almost does not occur in practice. Therefore (19) is the differential equation of $f(x)$ when $\varphi(X)$ is given.

When $\varphi(X)$ is linear with respect to X, we get $\varphi'(X) = const = A$ say, and from the equation (19), we get

$$v_2 + V_2 f'(x) - A v_2 v_3 \sqrt{1 + f'(x)^2} = 0 \quad \cdots\cdots \quad (23)$$

and solving this equation with respect to $f'(x)$ gives $f'(x) = const = a$ say, therefore $f(x) = ax + b$ where b is some constant. Consequently, if the travelling-time curve is a straight line, the curve of separation is also a straight line too. Of course some conditions must be satisfied so as to that the equation (23) has at least one solution. The calculation shows that the necessary and sufficient condition is

$$A^2 v_2^2 \leqq 1 \quad \text{and} \quad 0 \leqq V_2 + A v_2 v_3 \quad \cdots\cdots \quad (24).$$

Moreover when $V_2 < A v_2 v_3$, (23) has two solutions and when $V_2 \geqq A v_2 v_3$ only one solution.

6. Comments.

By my calculation, it is clear that in the case of two layers, if the curve of separation

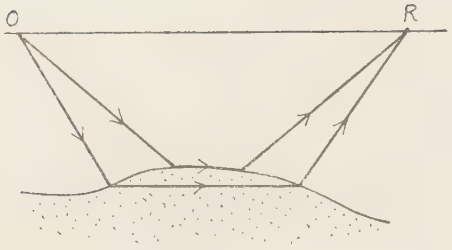

Fig. 2.

is a straight line, the travelling-time curve is also a straight line, and inversely, if the travelling-time curve is a straight line, the curve of separation is also a straight line, and when three layers, if two of the three curves (travelling-time curve and two curves of separation) are straight lines, the other is also straight line, but when only the linearity of $\varphi(X)$ is given, there is a freedom on the forms of $f(x)$ and $g(x)$ and none is introduced about them.

When $g''(x) > O$ or $f''(x) > O$, as Fig. 2 shows, there may occur some cases that a wave-ray enters from the first layer to the second layer and then second to first etc., and in practice, the estimation of the wave-ray which is in our question is very difficult.

The formulas (14) *and* (15) *when three layers,* (19) *and* (20) *when two layers are fundamental relations and we can introduce many problems from them.*

References

(1) See any Book written on the Seismic Prospecting.
(2) R. Adachi: On a Proof of Fundamental Formula Concerning Refraction Method of Geophysical Prospecting and Some Remarks. Kumamoto Journal of Science Vol. 2, No. 1.

ON THE NEWTON'S METHOD FOR THE APPROXIMATE SOLUTIONS OF SIMULTANEOUS EQUATIONS

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(Received November 30, 1955)

I tried to extend the Newton's approximate method for the solutions of a equation with one unknown to simultaneous equations, and have gotton a result. According this result, it is possible to separate the solutions and to estimate the limits of the errors of the solutions.

1. Introduction.

There often occurs to calculate the approximate solutions of simultaneous equations numerically. Concerning a equation $f(x)=0$ with one unknoun, there are many methods and as one of them, the **Newton's method** is usefull one and it is as follows.

Let x_0 be one root of the equation $f(x)=0$ and a, b be two values near x_0 and if $f'(a)\neq 0$, $f(a)\cdot f(b)<0$ and $f''(x)$ has always same sign with that of $f(a)$ when x is between a and b, then putting

$$a_1 = a - f(a)/f'(a)$$

gives $a<a_1<x_0$ or $x_0<a_1<a$,

that is, a_1 is better value than a. Next let

$$|f'(x)|>k, \quad k>|f(a)|\cdot q$$

then

$$|a_1-x_0|\leqq q\left\{|a_1-a|+q(a-b)^2\right\}^2.$$

By recurring this procedure, we can obtain the approximate value of the root at any accuracy. In the follwing I have tried to extend this method to simultaneous equations,

2. The Method.

Let (x_0, y_0) be a pair of the solutions of simultaneous equations

$$f(x, y)=0, \quad \varphi(x, y)=0 \qquad \cdots\cdots (1),$$

and in a domain neighbouring (x_0, y_0), let $f(x, y)$, $\varphi(x, y)$, f_x, f_y, φ_x, φ_y, f_{xx}, f_{xy}, f_{yy}, φ_{xx}, φ_{xy}, φ_{yy} be continuous. If (a, b') is in the neighbourhood of (x_0, y_0), we get $f(a, b')\fallingdotseq 0$, $\varphi(a, b')\fallingdotseq 0$. Now consider the equation $\varphi(a, y)=0$ with respect to y, then b' is an approximate solution of it and by the Newton's method or other suitable method concerning one unknown, we can calculate the solution at any accuracy. Consequently, if b is

one approximate solution and put $y_0 = b + \lambda$ then we can make $|\lambda|$ to be sufficiently small. Of course $f(a, b) \fallingdotseq 0$ as $b \fallingdotseq b'$.

Next, from $\varphi(x, y) = 0$, we can consider that y is a function of x and put

$$y = \Phi(x) \quad \cdots\cdots (2)$$

and by taking a neighbourhood of (x_0, y_0), we can consider that $\Phi(x)$ is one valued function of x except very special cases, therefore we get
$\Phi(x_0) = y_0$, $\Phi(a) = b$.
Moreover put

$$f(x, y) = f(x, \Phi(x)) = F(x) \quad \cdots\cdots (3)$$

then we have

$$F(a) = f(a, \Phi(a)) = f(a, b) \fallingdotseq 0$$
$$F(x_0) = f(x_0, \Phi(x_0)) = f(x_0, y_0) = 0.$$

Therefore if we put

$$x = a + \delta \quad \cdots\cdots (4),$$

$|\delta|$ is small value and

$$0 = F(x_0) = F(a+\delta) = F(a) + \delta F'(a) + \frac{\delta^2}{2} F''(a+\theta\delta) \quad \cdots\cdots (5)$$
$$0 < \theta < 1$$

and as the first approximation, put

$$F(a) + \delta_1 F'(a) = 0, \quad a_1 = a + \delta_1 = a - \frac{F(a)}{F'(a)} \quad \cdots\cdots (6)$$
$$\text{where } F'(a) \neq 0,$$

Then we have

$$(x_0 - a_1)(a_1 - a) = \frac{F(a) \cdot F''(a+\theta\delta)}{2\{F'(a)\}^2} \cdot \delta^2 \quad \cdots\cdots (7).$$

Therefore if $F''(x)$ always has the same sign with that of $F(a)$ in the considered domain, we get

$$(x_0 - a_1)(a_1 - a) > 0 \text{ that is } x_0 < a_1 < a \text{ or } a < a_1 < x_0 \quad \cdots\cdots (8).$$

Next, from $\varphi(x, y) = 0$ we can consider that x is a function of y and put

$$x = \Psi(y) \quad \cdots\cdots (9)$$

$$f(x, y) = f(\Psi(y), y) \quad G(y) \quad \cdots\cdots (10)$$

then, by similar method as above, we get

$$b_1 = b - \frac{G(b)}{G'(b)} \quad \cdots\cdots \quad (11)$$

$$(y_0 - b_1)(b_1 - b) = -\frac{G(b)\,G''(b+\theta'\varepsilon)}{2\{G'(b)\}^2}\varepsilon^2 \quad \cdots\cdots \quad (12)$$

where $y_0 = b + \varepsilon$ and $G'(b) \neq 0$. Therefore if $G''(b)$ has the same sign with that of $G(b)$, we get

$$y_0 < b_1 < b \quad \text{or} \quad b < b_1 < y_0 \quad \cdots\cdots \quad (13).$$

Next, if we put

$$\frac{d}{dx}\Phi(x) = y', \quad \frac{d^2}{dx^2}\Phi(x) = y''$$

then from $\varphi(x, y) = 0$, we get

$$y' = -\frac{\varphi_x}{\varphi_y}, \quad y'' = \frac{-1}{\varphi_y^3}\left(\varphi_{xx}\varphi_y^2 - 2\varphi_x\varphi_y\varphi_{xy} + \varphi_x^2\varphi_{yy}\right)$$

therefore we have

$$\left.\begin{aligned} F'(x) &= \frac{1}{\varphi_y}\left(f_x\varphi_y - f_y\varphi_x\right) \\ F''(x) &= \frac{1}{\varphi_y^3}\left(\varphi_y^2 f_{xx} - 2\varphi_x\varphi_y^2 f_{xy} + \varphi_x^2\varphi_y f_{yy} - f_y\varphi_y^2\varphi_{xx} + 2f_y\varphi_x\varphi_y\varphi_{xy} - f_y\varphi_y^2\varphi_{yy}\right) \end{aligned}\right\} \quad \cdots\cdots \quad (14).$$

Similarly

$$\left.\begin{aligned} G'(y) &= \frac{1}{\varphi_x}\left(f_y\varphi_x - f_x\varphi_y\right) \\ G''(y) &= \frac{1}{\varphi_x^3}\left(\varphi_x^2 f_{yy} - 2\varphi_x^2\varphi_y f_{xy} + \varphi_x\varphi_y^2 f_{xx} - f_x\varphi_x^2\varphi_{yy} + 2f_x\varphi_x\varphi_y\varphi_{xy} - f_x\varphi_y^2\varphi_{xx}\right) \end{aligned}\right\} \quad \cdots\cdots \quad (15)$$

also we have

$$\left.\begin{aligned} F(a) &= f\left(a, \varphi(a)\right) = f(a, b) \\ G(b) &= f\left(\Psi(b), b\right) = f(a, b) \end{aligned}\right\} \quad \cdots\cdots \quad (16)$$

Consequently, we have following method. At first, find two sets of values (α_1, β_1), (α_2, β_2) which satisfy

$$f(\alpha_1, \beta_1) > 0, \quad \varphi(\alpha_1, \beta_1) = 0; \quad f(\alpha_2, \beta_2) < 0, \quad \varphi(\alpha_2, \beta_2) = 0 \quad \cdots\cdots \quad (17)$$

then at least there is a pair (x_0, y_0) of the solutions of the equations in the domain $\mathcal{D}$ which is represented in Fig. 1. Of course these are always possible as $f(x, y)$ is continuous.

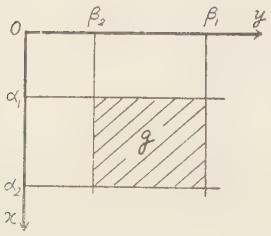

Fig. 1.

Now let $F''(x)$, $G''(y)$ be continuous and $\neq 0$ in the domain $\mathcal{D}$, then we can consider that $F''(x)$, $G''(y)$ have always same sign respectively by taking that $|\alpha_1-\alpha_2|$, $|\beta_1-\beta_2|$ to be sufficiently small.

Thus, we put to

$$a=\alpha_1,\quad b=\beta_1 \quad \text{when} \quad F''(x)>0$$

$$a=\alpha_2,\quad b=\beta_2 \quad \text{when} \quad F''(x)<0$$

and

$$a_1=a-\frac{F(a)}{F'(a)}=a-\frac{f(a,b)\,\varphi_y(a,b)}{f_x(a,b)\,\varphi_y(a,b)-f_y(a,b)\,\varphi_x(a,b)} \qquad (18).$$

Similarly,

$$\bar{a}=\alpha_1,\quad \bar{b}=\beta_1 \quad \text{when} \quad G''(y)>0$$

$$a=\alpha_2,\quad \bar{b}=\beta_2 \quad \text{when} \quad G''(y)<0$$

and

$$b_1=\bar{b}-\frac{G(b)}{G'(b)}=\bar{b}-\frac{f(\bar{a},b)\varphi_x(a,b)}{f_y(\bar{a},b)\,\varphi_x(\bar{a},\bar{b})-f_x(\bar{a},\bar{b})\,\varphi_y(\bar{a},b)} \qquad (19)$$

then (a_1,b_1) is more precise solutions than (a,b). Recurring this procedure we have (a_2,b_2), (a_3,b_3), ...

Next, if k,k are upper bounds of $|F''(x)|$, $|G''(y)|$ in the domain $\mathcal{D}$ respectively, and put

$$\frac{k}{2|F'(a)|}=q,\qquad \frac{k}{2|G'(b)|}=q \qquad (20)$$

then

$$|a_1-x_0|=\left|-\frac{F(a)}{F'(a)}-\delta\right|=\left|\frac{F''(a+\theta\delta)}{2F'(a)}\right|\cdot\delta^2\leqq q(-a+x_0)^2$$

$$-q(a_1-a+x_0-a_1)^2\leqq q\{|a_1-a|+q(x_0-a)^2\}^2\leqq q\{|a_1-a|+q(\alpha_1-\alpha_2)^2\}^2$$

$$\therefore\quad |a_1-x_0|\leqq q\{|a_1-a|+q(\alpha_1-\alpha_2)^2\}^2 \qquad (21),$$

similarly

$$|b_1-y|\leqq q\{|b_1-b|+q(\beta_1-\beta_2)^2\}^2 \qquad (22).$$

The geometrical meaning of this method is as follows. As shown in Fig. 2, let the intersection of the surface $z=f(x,y)$ and the cylinder $\varphi(x,y)=0$ be l, then the intersection of

the line l and the xy-plane is $(x_0, y_0, 0)$, and the curves $z = F(x)$ on the xz-plane, $z = G(y)$ on the yz-plane are the orthogonal projections of the line l to the xz-plane, the yz-plane respectively and above method means that we find a_1, b_1 by applying the Newton's method to these curves when one unknown.

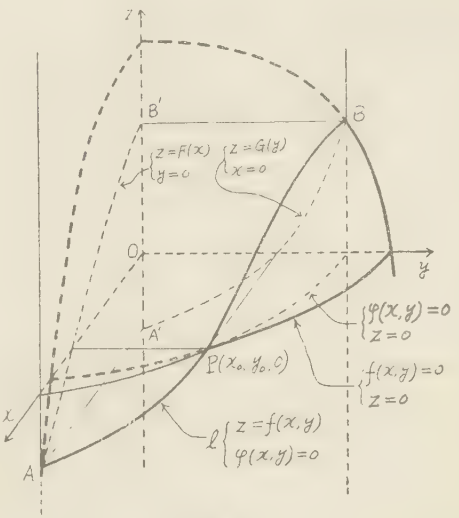

FIG. 2.

3. Examples.

EX. 1. The simultaneous equations

$$f(x,y) \equiv x^2 + y^2 - 9 = 0$$

$$\varphi(x,y) \equiv 1.69(x+1)^2 + 4(y-0.5)^2 - 27.04 = 0$$

have a pair of roots near $x \fallingdotseq 2$, $y \fallingdotseq 2$. Find more precise values of them.

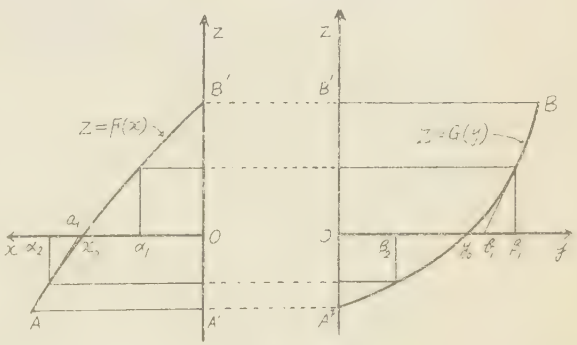

FIG. 3.

Solution:

At first we have

$$f_x = 2x, \quad f_y = 2y,$$

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 0$$

$$\varphi_x = 3.38(x+1),$$

$$\varphi_y = 4(2y-1), \quad \varphi_{xx} = 3.38, \quad \varphi_{yy} = 8, \quad \varphi_{xy} = 0$$

then

$$F(x) = x^2 + y^2 - 9, \quad F'(x) = \frac{2x(2y-1) - 1.69y(x+1)}{2y - 1}$$

$$F''(x) = \frac{1}{(2y-1)^3}\left\{(2y-1)^2(2.31y-2) - \frac{1}{2} \times (1.69)^2(x+1)^2\right\}$$

$$G(y) = x^2 + y^2 - 9, \quad G'(y) = \frac{3.38y(x+1) - 4x(2y-1)}{1.69(x+1)}$$

$$G''(y) = \frac{1}{(3.38)^2(x+1)^3}\left\{2(8y-4)^2 - 6.76(4.62x - 3.38)(x+1)^2\right.$$

Next, from $\varphi(2.09, \beta) = 0$ we get

$$(\beta - 0.5)^2 = \frac{1}{4}\left\{27.04 - 1.69 \times (3.09)^2\right\} = 2.725\ 928$$

$$\therefore \quad \beta = 2.151\ 038\ 5$$

and

$$f(2.09,\ \beta) = -\ 0.004\ 934 < 0$$

similarly, from $\varphi(2.10,\ \beta) = 0$ we get

$$\beta = 2.143\ 099\ 2$$

and

$$f(2.10,\ \beta) = 0.002\ 874 > 0$$

therefore we put

$$\left.\begin{aligned} \alpha_1 &= 2.100\ 000\ 0 \\ \beta_1 &= 2.143\ 099\ 2 \end{aligned}\right) \qquad \left.\begin{aligned} \alpha_2 &= 2.090\ 000\ 0 \\ \beta_2 &= 2.151\ 038\ 5 \end{aligned}\right).$$

When $2.09 < x < 2.10$, $2.143\ 099\ 2 < y < 2.151\ 038\ 5$
we get

$$F''(x) > \frac{1}{(2y-1)^3}\left\{3^2 \times 2.62 - \frac{1}{2} \times (1.69 \times 3.1)^2\right\} > 0$$

$$G''(y) < \frac{1}{(3.38)^3 (x+1)^3}\left\{2(3.38^2 - 2.16 - 4)^2 - 6.76(4.62 \times 2.09 - 3.38) \times (3.09)^2\right\} < 0$$

therefore

$$\left.\begin{aligned} a &= \alpha_1 = 2.100\ 000\ 0 \\ b &= \beta_1 = 2.143\ 099\ 2 \end{aligned}\right) \qquad \left.\begin{aligned} a &= \alpha_2 = 2.090\ 000\ 0 \\ \bar{b} &= \beta_2 = 2.151\ 038\ 5 \end{aligned}\right)$$

and

$$F(a) = 0.002\ 874, \qquad F'(a) = 0.783\ 378$$

$$\therefore \quad a_1 = a - \frac{F(a)}{F'(a)} = 2.100\ 000 - 0.003\ 669 = 2.096\ 331$$

$$G(\bar{b}) = -\ 0.004\ 934, \qquad G'(\bar{b}) = -\ 0.984\ 180$$

$$\therefore \quad b_1 = \bar{b} - \frac{G(b)}{G'(b)} = 2.151\ 038\ 5 - 0.005\ 013\ 3 = 2.146\ 025.$$

Moreover

$$|F''(x)| < \frac{1}{27}\left\{(2 \times 2.2 - 1)^2 (2.31 \times 2.2 - 2) - \frac{1}{2} \times (1.69)^2 \times 9\right\} \fallingdotseq 0.84$$

$$|G''(y)| < \frac{1}{(3.38)^2 \times 27} \left\{6.76(4.62 \times 2.1 - 3.38) \times (3.1)^2 - 2(8 \times 2 - 4)^2\right\} \fallingdotseq 0.41$$

therefore

$$q = \frac{0.84}{2 - 0.78} \fallingdotseq 0.54, \quad \bar{q} = \frac{0.41}{2 - 0.98} \fallingdotseq 0.21$$

and

$$|\alpha_1 - \alpha_2| = 0.01, \qquad |\beta_1 - \beta_2| \fallingdotseq 0.008$$

$$|a_1 - a| \fallingdotseq 0.0037, \quad |b_1 - b| \fallingdotseq 0.005$$

$$\therefore \quad |a_1 - x_0| < 0.54 \left\{0.0037 + 0.54 \times (0.01)^2\right\}^2 \fallingdotseq 0.000\ 008$$

$$|b_1 - y_0| < 0.21 \left\{0.005 + 0.21 \times (0.008)^2\right\}^2 \fallingdotseq 0.000\ 005.$$

Recurring,

from $\varphi(2.0963, \beta) = 0$ we have

$$\beta = 2.146\ 044\ 156\ 751$$

$$f(2.0963, \beta) = -0.000\ 020\ 794\ 826 < 0$$

from $\varphi(2.0964, \beta) = 0$ we have

$$\beta = 2.145\ 964\ 678\ 965$$

$$f(2.0964, \beta) = 0.000\ 057\ 363\ 365 > 0$$

therefore

$$\left.\begin{matrix}\alpha_1 = 2.096\ 4 \\ \beta_1 = 2.145\ 964\ 678\ 965\end{matrix}\right) \qquad \left.\begin{matrix}\alpha_2 = 2.096\ 3 \\ \beta_2 = 2.146\ 044\ 156\ 751\end{matrix}\right)$$

while $F''(x) > 0$, $G''(y) < 0$ therefore

$$a = \alpha_1, \quad b = \beta_1, \quad a = \alpha_2, \quad b = \beta_2$$

and

$$F(a) = 0.000\ 057\ 363\ 365, \qquad F'(a) = 0.781\ 532\ 11$$

$$G(b) = -0.000\ 020\ 794\ 826, \quad G'(b) = -0.983\ 308\ 90$$

$$\therefore \quad a_2 = a - \frac{F(a)}{F'(a)} = 2.0964 - 0.000\ 073\ 398\ 603 = 2.096\ 326\ 601\ 397$$

$$b_2 = b - \frac{G(b)}{G'(b)} = 2.146\ 044\ 156\ 751 - 0.000\ 021\ 147\ 807 = 2.146\ 023\ 008\ 944.$$

Next

$$|a_2 - a| \fallingdotseq 0.000\ 073, \quad |\alpha_1 - \alpha_2| = 0.0001$$

$$|b_2 - b| \fallingdotseq 0.000\ 021, \quad |\beta_1 - \beta_2| \fallingdotseq 0.000\ 08$$

and $q \fallingdotseq 0.54$, $\bar{q} \fallingdotseq 0.21$

$$\therefore \quad |a_2 - x_0| < 0.54\left\{0.000\ 073 + 0.54 \times (0.000\ \ 1)^2\right\}^2 \fallingdotseq 0.000\ 000\ 000\ 3$$

$$|b_2 - y_0| < 0.21\left\{0.000\ 021 + 0.21 \times (0.000\ 08)^2\right\}^2 \fallingdotseq 0.000\ 000\ 000\ 1$$

therefore

$$a_2 = 2.096\ 326\ 601\ 4$$

$$b^2 = 2.146\ 023\ 008\ 9$$

and the last figures of a_2, b_2 may have some errors.

EX. 2 Find the real roots of the simultaneous equations

$$f(x, y) \equiv 9x^2 + 16y^2 - 576 = 0$$

$$\varphi(x, y) \equiv (x-7)^2 + (y-5)^2 - (1.44)^2 = 0.$$

Solution:

The graphs of $f(x, y) = 0$, $\varphi(x, y) \doteqdot 0$ show that the required solutions lay near $x \fallingdotseq 6$, $y \fallingdotseq 4$.

We have

$$f_x = 18x, \quad f_y = 32y, \quad f_{xx} = 18, \quad f_{yy} = 32, \quad f_{xy} = 0$$

$$\varphi_x = 2(x-7), \quad \varphi_y = 2(y-5), \quad \varphi_{xx} = 2, \quad \varphi_{yy} = 2, \quad \varphi_{xy} = 0$$

and

$$F'(x) = \frac{224y - (14y+90)x}{y-5}, \qquad F''(x) = -\frac{160 \times 2.0736}{(y-5)^3} - 14$$

$$G'(y) = \frac{(14y+90)x - 224y}{x-7}, \qquad G''(y) = -\frac{126 \times 2.0736}{(x-7)^3} + 14.$$

From $\alpha = 5.9$, $\varphi(\alpha, \beta) = 0$ we have

$$(\beta - 5)^2 = 2.0736 - 1.2100 = (-0.929\ 30)^2$$

$$\therefore \quad \beta = 4.070\ 70 \quad \text{and} \quad f(\alpha, \beta) = 2.419\ 47 > 0$$

similarly from $\alpha = 6.0$, $\varphi(\alpha, \beta) = 0$,

$$\beta = 3.963\ 85, \quad f(\alpha, \beta) = -0.405\ 91 < 0$$

from $\alpha = 6.1$, $\varphi(\alpha, \beta) = 0$,

$$\beta = 3.875\ 90, \quad f(\alpha, \beta) = -0.748\ 34 < 0$$

from $\alpha = 6.2$, $\varphi(\alpha, \beta) = 0$,

$$\beta = 3.802\ 67, \quad f(\alpha, \beta) = 1.324\ 74 > 0.$$

It is easily shown that

$$F'(x) > 0 \quad \text{when} \quad 6.1 < x < 6.2 \quad \text{and} \quad F'(x) < 0 \quad \text{when} \quad 5.9 < x < 6.0,$$

therefore $F(x) = 0$ has each only one root in the intervals $(6.1, 6.2)$ and $(5.9, 6.0)$.

At first, we calculate the root which is in the interval $(5.9, 6.0)$,

from $\alpha = 5.95$, $\varphi(\alpha, \beta) = 0$ we get

$$\beta = 4.014\ 56, \text{ and } f(\alpha, \beta) = 0.489\ 05 > 0$$

from $\alpha = 5.96$, $\varphi(\alpha, \beta) = 0$

$$\beta = 4.004\ 01, \text{ and } f(\alpha, \beta) = 0.207\ 68 > 0$$

from $\alpha = 5.97$, $\varphi(\alpha, \beta) = 0$

$$\beta = 3.993\ 67, \text{ and } f(\alpha, \beta) = -0.041\ 49 < 0$$

from $\alpha = 5.969$, $\varphi(\alpha, \beta) = 0$

$$\beta = 3.994\ 694\ 57, \quad f(\alpha, \beta) = -0.017\ 995\ 64 < 0$$

from $\alpha = 5.968$, $\varphi(\alpha, \beta) = 0$

$$\beta = 3.995\ 721\ 16, \quad f(\alpha, \beta) = 0.005\ 817\ 44 > 0$$

and $F''(x) > 0$, $G''(y) > 0$ when $5.968 < x < 5.969$.

Consequently, we put

$$a = \bar{a} = 5.968, \quad b = \bar{b} = 3.995\ 721\ 16$$

then

$$F(a) = G(b) = 0.005\ 817\ 44$$

$$F'(a) = -23.968\ 488, \quad G'(b) = 23.324\ 657$$

$$\therefore \quad a_1 = a - \frac{F(a)}{F'(a)} = 5.968 + 0.000\ 242\ 712 = 5.968\ 242\ 712$$

$$b_1 = b - \frac{G(b)}{G'(b)} = 3.995\ 721\ 16 - 0.000\ 249\ 412 = 3.995\ 471\ 74.$$

Next, when $5.968 < x < 5.969$ we have $3.9957 \cdots\cdots > y > 3.994\ 6 \cdots\cdots$ and

$$|F''(x)| < \frac{160 \times 2.0736}{(1.004)^3} - 14 \fallingdotseq 317$$

$$|G''(y)| < \frac{126 \times 2.0736}{(1.031)^3} + 14 \fallingdotseq 250$$

$$\therefore \quad q \fallingdotseq \frac{317}{2 \cdot 24} \fallingdotseq 6.6, \quad \bar{q} \fallingdotseq \frac{250}{2 \cdot 23.3} \fallingdotseq 5.4$$

and

$$|a_1 - a| \fallingdotseq 0.000\ 24, \quad |\alpha_1 - \alpha_2| = 0.001$$

$$|b_1 - b| \fallingdotseq 0.000\ 25, \quad |\beta_1 - \beta_2| \fallingdotseq 0.001$$

$$\therefore \quad |a_1 - x_0| < 6.6 \left\{0.00024 + 6.6 \times (0.001)^2\right\}^2 \fallingdotseq 0.000\ 000\ 4$$

$$|b_1 - y_0| < 5.4 \left\{0.00025 + 5.4 \times (0.001)^2\right\}^2 \fallingdotseq 0.000\ 000\ 4$$

therefore $a_1 = 5.968\ 242\ 7, \quad b_1 \fallingdotseq 3.995\ 471\ 7$

and the last figures of a_1, b_1 may have some errors not greater than 4.

Similarly the roots which lay in the interval $(6.1, 6.2)$ is obtained as follows.

From $\alpha = 6.15, \quad \varphi(\alpha, \beta) = 0$ we have

$$\beta = 3.837\ 63, \quad f(\alpha, \beta) = 0.041\ 18 > 0$$

from 6.14, $\quad \varphi(\alpha, \beta) = 0$

$$\beta = 3.845\ 01, \quad f(\alpha, \beta) = -0.157\ 87 < 0$$

from $\alpha = 6.148, \quad \varphi(\alpha, \beta) = 0$

$$\beta = 3.839\ 096\ 90, \quad f(\alpha, \beta) = -0.000\ 223\ 84 < 0$$

from $\alpha = 6.1481, \quad \varphi(\alpha, \beta) = 0$

$$\beta = 3.839\ 023\ 53, \quad f(\alpha, \beta) = 0.001\ 829\ 11 > 0$$

and $F''(x) > 0, \quad G''(y) > 0$ when $6.1480 < x < 6.1481$

therefore we get

$$a = a = 6.1481, \quad b = \bar{b} = 3.839\ 023\ 53$$

and we have

$$a_1 = 6.148\ 010\ 87, \quad |a_1 - x_0| < 0.000\ 000\ 04$$

$$b_1 = 3.839\ 088\ 93, \quad |b_1 - y_0| < 0.000\ 000\ 03.$$

EX. 3 The simultaneous equations

$$f(x,y) \equiv \sqrt{15}\,(y-3) - x - \frac{1}{2}\sin 2\pi\,(\sqrt{15}\,x+y-3) = 0$$

$$\varphi(x,y) \equiv x^2 + (y-3)^2 - (2.55)^2 = 0$$

have a pair of roots near $x = 2.5$, $y = 3.7$. Find more precise values of them.

Solution:

By putting $\theta = 2\pi\,(\sqrt{15}x+y-3)$ we have

$$f_x = -\,(1+\sqrt{15}\,\pi\cos\theta),\quad f_y = \sqrt{15} - \pi\cos\theta,\quad \varphi_x = 2x,\quad \varphi_y = 2(y-3)$$

$$f_{xx} = 30\pi^2\sin\theta,\quad f_{xy} = 2\sqrt{15}\,\pi^2\sin\theta,\quad f_{yy} = 2\pi^2\sin\theta$$

$$\varphi_{xx} = \varphi_{yy} = 2,\quad \varphi_{xy} = 0$$

and

$$F'(x) = -\frac{\sqrt{15}\,x+y-3+\pi\left\{\sqrt{15}(y-3)-x\right\}\cos\theta}{y-3}$$

$$G'(y) = \frac{\sqrt{15}\,x+y-3+\pi\left\{\sqrt{15}(y-3)-x\right\}\cos\theta}{x}$$

$$F''(x) = \frac{1}{(y-3)^3}\left[2\left\{\sqrt{15}\,(y-3)-x\right\}^2(y-3)\pi^2\sin\theta - (2.55)^2\left(\sqrt{15} - \pi\cos\theta\right)\right]$$

$$G''(y) = \frac{1}{x^3}\left[2\left\{\sqrt{15}\,(y-3)-x\right\}^2 x\pi^2\sin\theta + (2.55)^2\left(1+\sqrt{15}\,\pi\cos\theta\right)\right].$$

From $\alpha = 2.5$, $\varphi(\alpha,\beta) = 0$ we have

$$\beta = 3.502\ 49,\quad \theta = 20\pi + 66°35'$$

$$\therefore\ f(\alpha,\beta) = -\,1.012\ 67 < 0$$

from $\alpha = 2.4$, $\varphi(\alpha,\beta) = 0$,

$$\beta = 3.861\ 68,\quad \theta = 20\pi + 56°28'$$

$$\therefore\ f(\alpha,\beta) = 0.556\ 11 > 0$$

therefore x_0 lays in the interval $(2.4, 2.5)$.

From $\alpha = 2.44$, $\varphi(\alpha,\beta) = 0$ we have

$$\beta = 3.740\ 877\ 9,\quad \theta = 20\pi + 68°44.67'$$

$$\therefore\ f(\alpha,\beta) = -\,0.036\ 579 < 0$$

from $\alpha = 2.43$, $\varphi(\alpha,\beta) = 0$,

$$\beta = 3.773\ 045\ 9,\quad \theta = 20\pi + 66°22.94'$$

$$\therefore \quad f(\alpha, \beta) = 0.015\ 874 > 0$$

While $F''(x) < 0, \quad G''(y) > 0$ when $2.43 < x < 2.44$

therefore, by putting

$$\left.\begin{aligned} a &= 2.44 \\ b &= 3.740\ 877\ 9 \end{aligned}\right) \qquad \left.\begin{aligned} \bar{a} &= 2.43 \\ b &= 3.773\ 045\ 9 \end{aligned}\right)$$

we have

$$F(a) = -0.036\ 579, \quad F'(a) = -14.4153$$

$$\therefore \quad a_1 = a - \frac{F(a)}{F'(a)} = 2.44 - 0.002\ 537\ 5 = 2.437\ 462\ 5$$

$$G(\bar{b}) = 0.105\ 874, \quad G'(b) = 4.483\ 23$$

$$\therefore \quad b_1 = b - \frac{G(b)}{G'(\bar{b})} = 3.773\ 046 - 0.023\ 616 = 3.749\ 430.$$

Next

$$|F''(x)| = \frac{1}{(y-3)^3}\left[(2.55)^2(\sqrt{15}-\pi\cos\theta) - 2\left\{\sqrt{15}(y-3)-x\right\}^2(y-3)\pi^2\sin\theta\right]$$

$$< \frac{1}{(y-3)^3}\left[(2.55)^2(\sqrt{15}-\pi\times 0.363) - 2(\sqrt{15}\times 0.73 - 2.44)^2\times 0.73\pi^2\times 0.91\right] < 40$$

similarly

$$|G'(y)| < 3.4,$$

therefore we get

$$q \fallingdotseq \frac{10}{2\times 14.4} \fallingdotseq 1.4, \quad q = \frac{3.4}{2\times 4.5} \fallingdotseq 0.38$$

and

$$|a_1 - a| \fallingdotseq 0.0025, \quad |\alpha_1 - \alpha_2| = 0.01$$

$$|b_1 - \bar{b}| \fallingdotseq 0.024, \quad |\beta_1 - \beta_2| \fallingdotseq 0.032$$

$$\therefore \quad |a_1 - x_0| < 1.4\left\{0.0025 + 1.4\times 0.0001\right\}^2 \fallingdotseq 0.000\ 01$$

$$|b_1 - y_0| < 0.38\left\{0.024 + 0.38\times(0.032)^2\right\}^2 \fallingdotseq 0.000\ 2$$

therefore

$$a_1 = 2.437\ 46, \quad b_1 = 3.749\ 4$$

and the last figures of a_1, b_1 may have some errors.

Of course we can find the value of b_1 from $\varphi(a_1, b_1) = 0$, by putting $a_1 = 2.437\ 46$ and we get the following result

$$b_1 = 3.749\ 19 \quad \text{and} \quad |b_1 - y_0| < 0.000\ 03.$$

4. Remarks.

Occasionally, we employ the following procedure which is ordinarily called **Newton's method** or **Newton Raphson's** method, that is

" Let (a, b) be a set of approximate values of (x_0, y_0) which is one set of the solutions of the simultaneous equations

$$f(x, y) = 0, \quad \varphi(x, y) = 0$$

and put

$$x_0 = a + \delta, \quad y_0 = b + \varepsilon$$

then

$$f(a+\delta, b+\varepsilon) = 0, \quad \varphi(a+\delta, b+\varepsilon) = 0$$

and we have following approximate equations

$$f(a, b) + \delta f_x(a, b) + \varepsilon f_y(a, b) = 0$$

$$\varphi(a, b) + \delta \varphi_x(a, b) + \varepsilon \varphi_y(a, b) = 0.$$

Let δ_1, ε_1 be the solutions of these equations and put to $a_1 = a + \delta_1$, $b_1 = b + \varepsilon_1$ as more precise values of x_0, y_0 than a_1, b. "

But *this procedure is not rigorous*, for instance in the simultaneous equations which are shown in EX. 2, $a = 6.0$, $b = 3.9$ are one pair of approximate values of the solutions and

$$f(a, b) = -8.6400, \quad \varphi(a, b) = 0.1364, \quad f_x(a, b) = 108$$

$$f_y(a, b) = 124.8, \quad \varphi_x(a, b) = -2, \quad \varphi_y(a, b) = -2.2,$$

from the equations

$$108\delta_1 + 124.8\varepsilon_1 = 8.6400$$

$$-2\delta_1 - 2.2\varepsilon_1 = -0.1364,$$

we get $\delta_1 = -0.1654$, $\varepsilon_1 = 0.2124$ $\therefore$ $a_1 = 5.8346$, $b_1 = 4.1124$,

but these are more awkward values than (a, b).

Next consider the equations

$$f(x, y) \equiv y - \sin(x+y) = 0 \qquad 0 \leqq x \leqq \pi$$

$$\varphi(x, y) \equiv (x-0.6)^2 + (y+1.8)^2 - (2.7)^2 = 0 \qquad 0 \leqq y,$$

then

$$f\,(1.65, 0.70) = -\,0.01147 \fallingdotseq 0, \quad \varphi\,(1.65, 0.70) = 0.0625 \fallingdotseq 0$$

therefore (1.65, 0.70) seem to be one pair of approximate solutions of the equations, and putting $a = 1.65$, $b = 0.70$ we have

$$f = -\,0.011\,47, \quad \varphi = 0.0625, \quad f_x = 0.7027$$

$$f_y = 1.702\,7, \qquad \varphi_x = 2.10, \qquad \varphi_y = 5.0,$$

and from

$$0.7027\delta_1 + 1.7027\varepsilon_1 = 0.011\,47$$

$$2.1\quad \delta_1 + 5.0\quad \varepsilon_1 = -\,0.0625$$

we get $\delta_1 = -\,0.789$, $\varepsilon_1 = 0.319$

therefore $a_1 = 0.861$, $b_1 = 1.019$.

Recurring this method we have $a_2 = 1.493$, $b_2 = 0.619$ and so on, but a, a_1, a_2, ……; b, b_1, b_2, …… seem to do not converge. As a matter of fact the simultaneous equations do not possess a pair of solutions near (1.65, 0.70). *In practice, so-called Newton's method is very convenient one and in most cases it gives good result, and only in particular cases it gives awkward result. It has a defect in the point that the estimation of the limits of the errors is difficult.*

On the other hand, by our method we can estimate not only the existence of the solutions but also the limits of the errors of the solutions, but it has a defect in the point that the calculation is considerably troublesome.

As explained in EX. 3, we may calculate the value of b_1 from the equation $\varphi(a_1, b_1) = 0$ or $f(a_1, b_1) = 0$ after a_1 has been determined.

It is no difficult to extend this method to a case of many number of unknowns.

THE EFFECT OF THE INTERNAL STRESS ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD (II)

Shigeo MATSUMAE

(Received November 10, 1955)

[I] Introduction

In the previous paper,[1] we studied the relation between the longitudinal magnetostriction of nickel in weak magnetic field and the compressive stress in the direction of the applied magnetic field, which was obtained when the compressive stresses of various magnitudes were applied individually to each specimen which had the same components and was annealed in the same manner.

Now, we study the change of the longitudinal magnetostriction in weak magnetic field, when the compressive stress, applied to one and the same specimen in the direction of the applied magnetic field, increases stepwise. Experimental results are as follows.

[II] Experimental results and Discussions

Specimen, Experimental apparatus, and method of measurement are the same as in the previous paper.[1]

The specimen is annealed at 900°C for 90 minutes, cooled to the room temperature, and then demagnetized by the alternate magnetic field having the maximum amplitude of about 0.7 Oersted. As described in the previous paper, the longitudinal magnetostriction in weak magneitc field is so small as to be unobservable, when the specimen is not under a compressive stress.

When the compressive stress of 0.1kg/mm² is applied to the specimen in the direction of the applied magnetic field, the result of the measurement is shown in Fig. 1 and Fig. 2. After this measurement, the specimen is demagnetized again by the alternate magnetic field having the maximum amplitude of 0.7 Oersted and then the compressive stress of 0.2kg/mm² is applied furthermore to the specimen, in addition to the compressive stress of 0.1kg/mm² which is applied already to the specimen. The experimental result in this case is also shown in Fig. 1 and Fig. 2. The above process being repeated, we take measurement of the longitudinal magnetostriction of the same specimen in weak magnetic field under the compressive stresses of 0.1kg/mm², 0.3kg/mm², 0.5kg/mm², 0.7kg/mm², 0.9kg/mm², and 1.2kg/mm², step by step. These experimental results are shown in Fig. 1 and Fig. 2.

The experimental results show such a tendency that the longitudinal magnetostriction in weak magnetic field increases at first, reaches to a maximum value and, then, decreases so small as to be unobservable, with the increase of the compressive stress, as in the previous

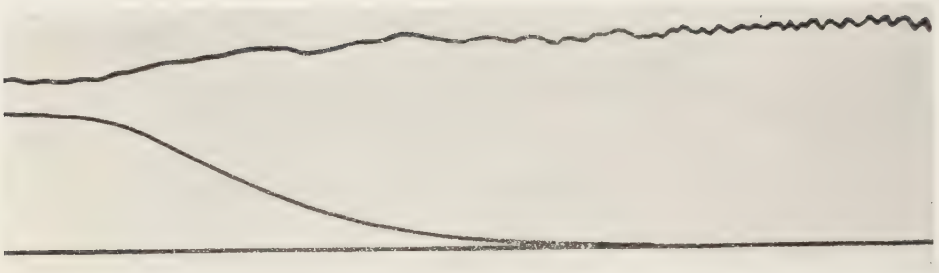

Fig. 1 $\sigma = 0.1\text{kg/mm}^2$

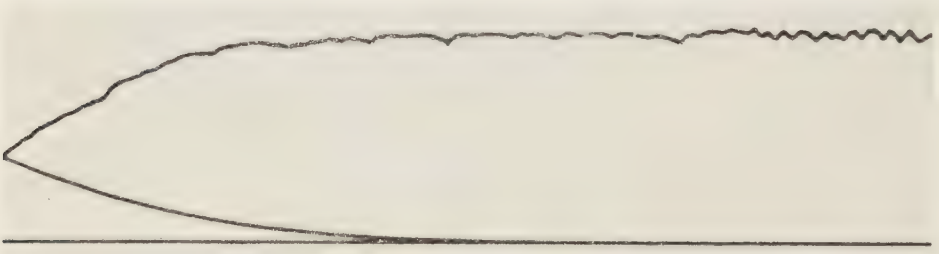

Fig. 1 $\sigma = 0.3\text{kg/mm}^2$

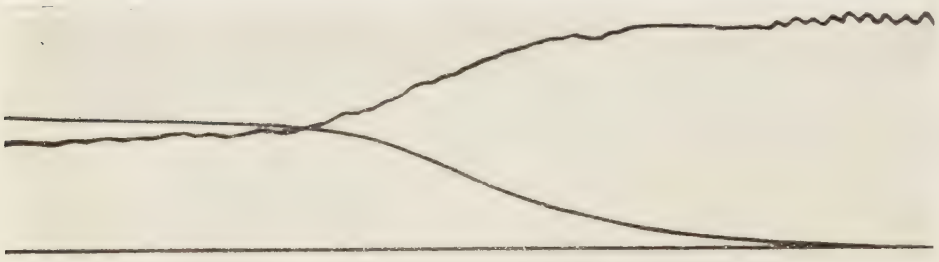

Fig. 1 $\sigma = 0.5\text{kg/mm}^2$

Fig. 1 $\sigma = 0.7\text{kg/mm}^2$

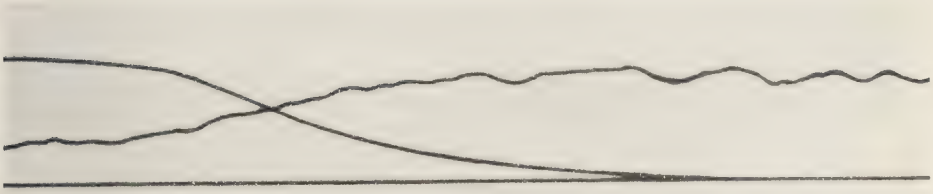

Fig. 1 $\sigma = 0.9 kg/mm^2$

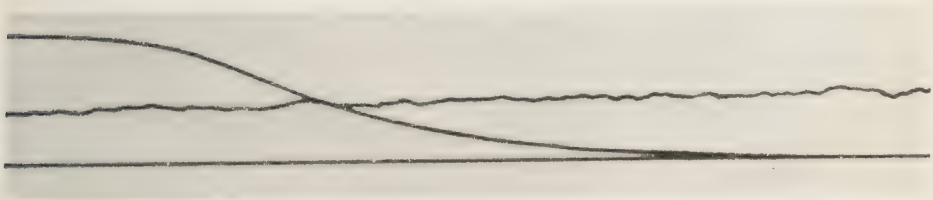

Fig. 1 $\sigma = 1.2 kg/mm^2$

Fig. 1. Oscillograph-records of the longitudinal magnetostriction of nickel in weak magnetic field when the compressive stress, applied to the specimen, increases step by step.

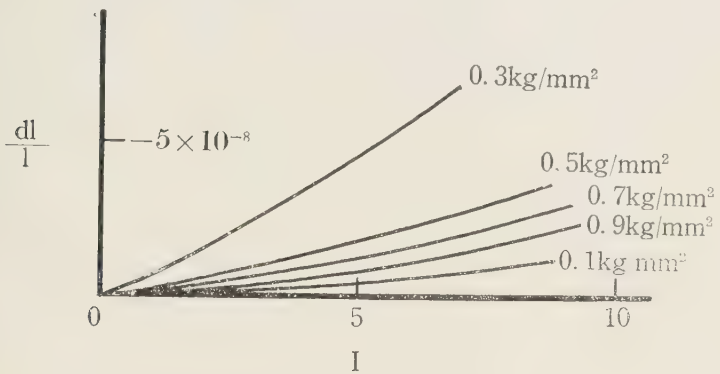

Fig. 2

Fig. 2. $\frac{dl}{l}$ —I curves of the above oscillograph-records.

paper. But the longitudinal magnetostriction in weak magnetic field decreases more rapidly with the increase of the compressive stress in this case than in the previous paper, in which the compressive stresses of different magnitudes are individually applied to the specimens which have the same components and are annealed in the same manner. This seems to be caused chiefly by the fact, whether the specimen is demagnetized at several times by a alternate magnetic field or not, in the course of applying the compressive stress to the specimen, as the magnetic domains about right angle to the magnetizing field are considered to diminish by demagnetizing the specimen with a alternate magnetic field.

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QUASI-STATIONARY PROCESS (III)
CHANGES OF STATE OF GAS BY THE SLOW CHANGES OF ITS BOUNDARY CONDITIONS

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(Received November 30, 1955)

As the first step to study the influences of the variations of the boundary upon the changes of states of systems, we have studied the changes of a simple ideal gas system produced by the various given changes of the boundary conditions.

1. Introduction

The state of a system will sometimes be changed by itself even though its boundary maintains a constant state. Also, when a system is in an equilibrium state, the change of the state of the boundary causes the change of the state of the system. In this work we study the slow changes of a gas system caused by the slow changes of the boundary when the gas system is in an equilibrium state or in a slightly different state from equilibrium.

When a system is in a stationary state, the state of the boundary is generally stationary. The slight change of the boundary causes the slight change of the system and we have already considered a few problems in the previous papers, belonging to this case. We want to treat the general slow processes caused by the slow changes of the boundary, and in this paper we treat, in the first place, the slow processes of a simple gas system.

We take unit mass of any ideal gas, whose temperature, pressure, and volume are expressed by T, p and v respectively. The temperature of the boundary is T_1, and it is assumed that the heat is transferred per unit time from the boundary to the gas in the form $k(T_1 - T)$, where k is a constant, relating to the conduction and radiation of heat. If in a small time dt, the temperature and the volume accept some small changes dT and dv respectively, and all the processes proceed very slowly, we can express the energy relation in the following expression

$$k(T_1 - T)\,dt = pdv + c_v\,dT, \qquad (1)$$

where c_v is the specific heat at constant volume of the gas. Putting the relations $pv = R'T$ and $pdv = -vdp + R'dT$ in (1), we get the expressions

$$k(T_1 - T)\,dt = -vdp + c_p\,dT, \qquad (1')$$

$$k\left(\frac{T_1}{T} - 1\right) = R'\frac{dlogv}{dt} + c_v\frac{dlogT}{dt} \qquad (2)$$

and

$$k\left(\frac{T_1}{T}-1\right)=-R'\frac{d\log p}{dt}+c_p\frac{d\log T}{dt}, \qquad \cdots\cdots (2')$$

where R is the specific gas constant and C_p is the specific heat at constant pressure of the gas.

2. The Changes of Gas by the Changes of the Boundary whose Temperature is kept Constant. Initial Temperature is Equal to the Temperature of the Boundary.

We take unit mass of the gas which is initially in an equilibrium state under the constant boundary state. If the state of the boundary of the gas happens to change slowly, the state of the gas will be changed slowly. The initial temperature of the gas which is equal to the temperature of the boundary, is expressed by T_{10}. We shall consider the various changes which are caused by the various given changes of the boundary in the following.

(i) By the slow change of the boundary, the volume of the gas is changed according to the expression $v=v_0e^{bt}$, where v_0 is the initial volume of the gas and b is a constant. In this case the expressions (1) and (2) become

$$k(T_{10}-T)=R'bT+c_v\frac{dT}{dt} \qquad \cdots\cdots (3)$$

and

$$k\left(\frac{T_{10}}{T}-1\right)=R'b+c_v\frac{d\log T}{dt} \qquad \cdots\cdots (4)$$

respectively. Frome (3) we get

$$kT_{10}-(k+R'b)T=c_v\frac{dT}{dt}.$$

Integrating this, we get

$$T-\frac{k}{k+R'b}T_{10}=Ae^{-\frac{k+R'b}{c_v}t} \qquad \cdots\cdots (5)$$

where A is the integration constant. As the initial temperature of the gas is T_{10}, A must take the value $A=T_{10}\frac{R'b}{k+R'b}$.

Accordingly, we get the solution of (3) or (4) in the following form.

$$T=\frac{T_{10}}{k+R'b}\left\{k+R'be^{-\frac{k+R'b}{c_v}t}\right\}. \qquad \cdots\cdots (5')$$

Thus the temperature of the gas slowly changes from the initial value T_{10} to the final value $T_{10}\frac{k}{k+R'b}$.

As the changes of the volume and the temperature of the gas are given, the change of the pressure p of the gas is given by

$$p = \frac{R'T}{v} = \frac{R'}{v_0}\frac{T_{10}}{k+R'b}\left(k + R'be^{-\frac{k+R'b}{c_v}t}\right)e^{-bt}.$$

If the initial pressure of the gas is p_0 which is equal to $\frac{RT_{10}}{v_0}$, the above expression becomes

$$p = \frac{p_0}{k+R'b}\left(ke^{-bt} + R'be^{-\frac{k+c_pb}{c_v}t}\right) \quad \cdots\cdots (6)$$

or

$$p = \frac{p_0}{k+R'b}\left(ke^{-bt} + R'be^{-\left(\frac{k}{c_v}+b\gamma\right)t}\right), \quad \cdots\cdots (6')$$

where γ is the ratio of the two specific heats c_p and c_v. The pressure of the gas slowly changes from the initial value $p_0 = \frac{RT_{10}}{v_0}$ to the final value 0, if b is positive.

(ii) By the slow change of the boundary, the work rate of the gas is maintained to be constant, i. e., $p\frac{dv}{dt} = a$, where a is a constant.

Putting the given condition $p\frac{dv}{dt} = a$ in (1), we obtain the following expression,

$$k(T_{10} - T) = a + c_v\frac{dT}{dt}. \quad \cdots\cdots (7)$$

Integrating this equation, we get

$$T - T_{10} + \frac{a}{k} = Ae^{-\frac{k}{c_v}t}, \quad \cdots\cdots (8)$$

where A is the integration constant. A is to be determined by the initial condition that at $t = 0$, $T = T_{10}$, and we know that A must take the value $-\frac{a}{k}$, and so (8) becomes

$$T = T_{10} - \frac{a}{k}\left(1 - e^{-\frac{k}{c_v}t}\right). \quad \cdots\cdots (9)$$

Thus we know that T changes slowly from the initial value T_{10} to the final value $T_{10} - \frac{a}{k}$.

The time variations of volume and pressure are obtained in the following way. Putting the relation $p = \frac{R'T}{v}$ in $p\frac{dv}{dt} = a$, we obtain

$$\frac{d\log v}{dt} = \frac{a}{R'T} = \frac{a}{R'\left\{T_{10} - \frac{a}{k}\left(1 - e^{-\frac{k}{c_v}t}\right)\right\}}. \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (10)$$

Also, from $pv = R'T$, we obtain

$$\frac{d\log p}{dt} = \frac{d\log T}{dt} - \frac{d\log v}{dt} = \frac{1}{T}\frac{dT}{dt} - \frac{a}{R'T} = -\frac{1}{T}\left(\frac{a}{R'} - \frac{dT}{dt}\right),$$

and as we can obtain from (9) the relation

$$\frac{dT}{dt} = -\frac{a}{c_v}e^{-\frac{k}{c_v}t},$$

the relation

$$\frac{d\log p}{dt} = -\frac{a\left(\frac{1}{R'} + \frac{1}{c_v}e^{-\frac{k}{c_v}t}\right)}{T_{10} - \frac{a}{k}\left(1 - e^{-\frac{k}{c_v}t}\right)} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (11)$$

is resulted.

(iii) By the slow change of the boundary, the pressure of the gas is changed according to the relation $p = p_0 e^{-ct}$, where c is some constant.

Putting the relation $\frac{d\log p}{dt} = -c$ in (2), and considering the relation $T_1 = T_{10}$ in this case, we get

$$k(T_{10} - T) = R'cT + c_p\frac{dT}{dt} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (12)$$

or

$$kT_{10} - (k + R'c)T = c_p\frac{dT}{dt}.$$

Integrating this, we get

$$T = \frac{k}{k + R'c}T_{10} - Ae^{-\frac{k+R'c}{c_p}t}, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (13)$$

where A is the integration constant. Consideriug the initial condition that at $t = 0$, $T = T_{10}$, we can determine A and obtain the solution

$$T = \frac{T_{10}}{k+R'c}\left(k + R'ce^{-\frac{k+R'c}{c_p}t}\right). \quad \cdots\cdots (13')$$

Thus the temperature of the gas slowly changes from the initial value T_0 to the final value $T_{10}\frac{k}{k+R'c}$.

The change of the volume is obtained as follows,

$$v = \frac{R'T}{p} = R'\frac{T_{10}}{p_0}\frac{1}{k+R'c}\left\{ke^{ct} + R'ce^{-\left(\frac{k+R'c}{c_p}-c\right)t}\right\}$$

or

$$v = \frac{v_0}{k+R'c}\left\{ke^{ct} + R'ce^{\left(\frac{c}{\gamma}-\frac{k}{c_p}\right)t}\right\}, \quad \gamma = \frac{c_p}{c_v}. \quad \cdots\cdots (14)$$

3. The Changes of Gas by the Changes of the Boundary whose Temperature is kept Constant. The Initial temperature is slightly Different from the Temperature of the Boundary.

(i) The case where the volume of the gas slowly changes according as $v = v_0 e^{ct}$ by the change of the boundary.

In this case, the temperature changes according as (5), but as the integration constant A must satisfy the relation,

$$A = T_0 - \frac{k}{k+R'b}T_{10},$$

where T_0 is the initial temperature of the gas, we can obtain the time variation of the temperature of the gas as follows,

$$T = \frac{k}{k+R'b}T_{10} + \left(T_0 - \frac{k}{k+R'b}T_{10}\right)e^{-\frac{k+R'b}{c_p}t}. \quad \cdots\cdots (15)$$

Thus the temperature of the gas slowly changes from the initial value T_0 to the final value $\frac{k}{k+R'b}T_{10}$, if $k + R'b > 0$.

Now, in this case, it is considered that at some time the temperature of the gas T may become equal to the temperature of the boundary T_{10}. The time t_1 at which T becomes T_{10} is obtained by using (15) as the following.

$$T_{10}\frac{R'b}{k+R'b} = \left(T_0 - \frac{k}{k+R'b}T_{10}\right)e^{-\frac{k+R'b}{c_p}t_1}$$

or

$$e^{\frac{k+R'b}{c_v}t_1} = \frac{k\triangle T_0 + R'bT_0}{T_{10}R'b} = \frac{T_0}{T_{10}} + \frac{\triangle T_0}{T_{10}}\frac{k}{R'b}. \quad \triangle T_0 = T_0 - T_{10}.$$

$$\therefore \quad t_1 = \frac{c_v}{k+R'b}\log\left\{1 + \frac{\triangle T_0}{T_{10}}\left(1 + \frac{k}{R'b}\right)\right\}. \qquad \cdots\cdots (16)$$

If both the values of $\frac{\triangle T_0}{T_{10}}$ and $\frac{k}{R'b}$ are very small compared to 1, (16) becomes

$$t_1 = \frac{c_v}{R'b}\left(1 - \frac{k}{R'b}\right)\frac{\triangle T_0}{T_{10}}. \qquad \cdots\cdots (16')$$

(ii) The case where the pressure of the gas slowly changes according to $p = p_0 e^{-\alpha t}$ by the change of the boundary.

In this case, the temperature changes according to (13), but the integration constant A must take the value

$$A = T_0 - \frac{k}{k+R'c}T_{10},$$

and the temperature changes according as

$$T = \frac{k}{k+R'c}T_{10} + \left(T_0 - \frac{kT_{10}}{k+R'c}\right)e^{-\frac{k+R'c}{c_p}t} \qquad \cdots\cdots (17)$$

In this case, also, the time t_1, at which T becomes epual to T_{10}, is obtained from (17) as the following.

$$T_{10}\frac{R'c}{k+R'c} = \frac{k\triangle T_0 + T_{10}R'c}{k+R'c}e^{-\frac{k+R'c}{c_p}t_1}.$$

$$t_1 = \frac{c_p}{k+R'c}\log\left\{1 + \frac{\triangle T_0}{T_{10}}\left(1 + \frac{k}{R'c}\right)\right\}. \qquad \cdots\cdots (18)$$

If both the values of $\frac{\triangle T_0}{T_{10}}$ and $\frac{k}{R'c}$ are very small compared to 1, (18) becomes

$$t_1 = \frac{c_p}{R'c}\left(1 - \frac{k}{R'c}\right)\frac{\triangle T_0}{T_{10}}. \qquad \cdots\cdots (18')$$

4. The Changes of Gas by the Changes of the Boundary whose Temperature T changes as $T = T_{10}e^{-\alpha t}$. The Initial Temperature of the Gas is Equal to that of the Boundary.

(i) By the slow change of the boundary, the volume of the gas is changed according

to $v = v_0 e^{bt}$.

Putting the relations $T_1 = T_{10} e^{-\alpha t}$, $\frac{d\log v}{dt} = b$ and $p = \frac{R'T}{v}$ in (1), we get

$$kT_{10} e^{-\alpha t} - (k + R'b) T = c_v \frac{dT}{dt} . \quad \cdots\cdots (19)$$

To solve this equation, we assume $T = Ae^{-\alpha t} + Be^{-\beta t}$, where A, B and β are some constants, and put this in (19), then we get the relation

$$\left\{kT_{10} - (k + R'b) A + c_v A\alpha\right\} e^{-\alpha t} = \left\{(k + R'b) B - c_v B\beta\right\} e^{-\beta t}$$

From this, the values of A and β are determined as follows.

$$A = \frac{kT_{10}}{k - R'b - c_v\alpha} , \quad \beta = \frac{k - R'b}{c_v} \quad \cdots\cdots (20)$$

The value of B is determined by the initial condition that at $t = 0$, T is equal to T_{10}. Thus we get

$$B = T_{10} - A = T_{10} \frac{R'b - c_v\alpha}{k + R'b - c_v\alpha} . \quad \cdots\cdots (21)$$

The time variation of the temperature T is, therefore, expressed by

$$T = \frac{T_{10}}{k + R'b - c_v\alpha} \left\{ke^{-\alpha t} + (R'b - c_v\alpha) e^{-\frac{k + R'b}{c_v} t}\right\} . \quad \cdots\cdots (22)$$

Thus the temperature T changes from the initial value T_{10} to the final value 0, if $\alpha > 0$ and $k + R'b > 0$.

The time variation of pressure is expressed by

$$p = \frac{R'T}{v} = \frac{R'T_{10}}{v_0 (k + R'b - c_v\alpha)} \left\{ke^{-(\alpha + b)t} + (R'b - c_v\alpha) e^{-\left(\frac{k + R'b + c_v b}{c_v}\right) t}\right\} \quad (23)$$

or

$$p = \frac{p_0}{k + R'b - c_v\alpha} \left\{ke^{-(\alpha + b)t} + (R'b - c_v\alpha) e^{-\left(\frac{k}{c_v} + b\gamma\right) t}\right\} , \quad \cdots\cdots (23')$$

where $p_0 = \frac{R'T_{10}}{v_0}$, $c_p = R' + c_v$ and $\gamma = \frac{c_p}{c_v}$.

Thus the pressure p changes from the initial value p_0 to the final value 0, if $\alpha + b > 0$ and $\frac{k}{c_v} + b\gamma > 0$.

In a special case where the relation $\alpha = \frac{R'b}{c_v}$ is satisfied, we get

$$T = T_{10}\, e^{-\frac{R'b}{c_v}t} \text{ and } p = p_0\, e^{-(\alpha+b)t} = p_0\, e^{-b_\gamma t}, \quad \cdots\cdots (24)$$

which are the values equal to those obtained by putting $k = 0$ in (22) and (23'). This case is the adiabatic one.

(ii) By the slow change of the boundary, the pressure of the gas is changed according to $p = p_0\, e^{-ct}$.

Putting the relations $T_1 = T_{10}\, e^{-\alpha t}$, $\frac{diogp}{dt} = -c$ and $v = \frac{R'T}{p}$ in (1'), we get

$$kT_{10}\, e^{-\alpha t} - (k + R'c)\, T = c_p \frac{dT}{dt}. \quad \cdots\cdots (25)$$

To solve this equation, we assume $T = Ae^{-\alpha t} + Be^{-\beta t}$ as in the case (i), and put this in (25), then we get the relation

$$\left\{kT_{10} - (k + R'c)\, A + c_p A\alpha\right\} e^{-\alpha t} = \left\{(k + R'c)\, B - c_p B\beta\right\} e^{-\beta t}.$$

From this, the values of A and β are determined as follows.

$$A = \frac{k\, T_{10}}{k + R'c - c_p\alpha}, \quad \beta = -\frac{k + R'c}{c_p}.$$

The value of B is determined by the initial condition as follows.

$$B = T_{10}\, \frac{R'c - c_p\, \alpha}{k + R'c - c_p\alpha}$$

Therefore, the solution of (25) is expressed by

$$T = \frac{T_{10}}{k + R'c - c_p\alpha}\left\{ke^{-\alpha t} + (R'c - c_p\, \alpha)\, e^{-\frac{k + R'c}{c_p}t}\right\}. \quad \cdots\cdots (26)$$

T changes slowly from the initial value T_{10} to the final value 0, when $\alpha > 0$ and $k + R_1 c > 0$.

The time variation of the volume v is expressed by

$$v = R'\frac{T}{p} = R'\frac{T_{10}}{p_0\,(k + R'c - c_p\alpha)}\left\{ke^{(c-\alpha)t} + (R'c - c_p\alpha)\, e^{\frac{c_p c - k - R'c}{c_p}t}\right\}$$

or

$$v = \frac{v_0}{k + R'c - c_p\alpha}\left\{ke^{(c-\alpha)t} + (R'c - c_p\, \alpha)\, e^{\left(\frac{c}{\gamma} - \frac{k}{c_p}\right)t}\right\}. \quad \cdots\cdots (27)$$

In a special case where the relation $\alpha = \frac{R'c}{c_p}$ is satisfied, (26) and (27) become

$$T = T_1 e^{-\alpha t} = T_1 e^{-\frac{R'c}{c_p} t}$$

and (28)

$$v = v_0 e^{(c-\alpha)t} = v_0 e^{\frac{c}{\tau} t}.$$

respectively, which are the values obtained by putting $k = 0$ in (26) and (27). This case is the adiabatic case.

5. The Changes of Gas by the Changes of the Boundary whose Temperature T_1 changes as $T_1 = T_{10} e^{-\alpha t}$. The Initial Temperature T_0 of the Gas is slightly Different from that of the Boundary.

(i) The case where the volume of the gas slowly changes according to $v = v_0 e^{bt}$ by the change of the boundary.

In this case the energy equation is (19), and the solution is obtained in the same way as in 4. (i). But the initial condition is that at $t = 0$, $T = T_0 (\neq T_{10})$.

The solution of (19), in this case, is obtained

$$T = k \frac{T_{10}}{k + R'b - c_v\alpha} e^{-\alpha t} + \left(T_0 - \frac{kT_{10}}{k + R'b - c_v\alpha} \right) e^{-\frac{k+R'b}{c_v} t} \quad \cdots\cdots \quad (29)$$

or

$$T = \frac{kT_{10}}{k + R'b - c_v\alpha} \left(e^{-\alpha t} - e^{-\frac{k+R'b}{c_v} t} \right) + T_0 e^{-\frac{k+R'b}{c_v} t} \quad \cdots\cdots\cdots \quad (29')$$

In a special case where $\frac{T_0}{T_{10}} = \frac{k}{k + R'b - c_v\alpha}$ is satisfied,

$$T = T_0 e^{-\alpha t}.$$

The time variation of p is obtained as follows.

$$p = \frac{R'T}{v} = R' \frac{kT_{10}}{v_0 (k + R'b - c_v\alpha)} \left\{ e^{-(b+\alpha)t} - e^{-\left(b + \frac{k+R'b}{c_v}\right) t} \right\}$$

$$+ \frac{R'T_0}{v_0} e^{-\left(b + \frac{k+R'b}{c_v}\right) t} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (30)$$

or

$$p = p_0 \frac{kT_{10}}{T_0(k+R'b-c_v\alpha)} \left\{ e^{-(b\gamma+\alpha)t} - e^{-\left(b\gamma+\frac{k}{c_v}\right)t} \right\} + p_0 e^{-\left(b\gamma+\frac{k}{c_v}\right)t} \quad \cdots \quad (30')$$

In a special case where the relation $\alpha = \frac{R'b}{c_v}$ is satisfied, (29) and (30′) become

$$T = T_{10}e^{-\alpha t} + (T_0 - T_{10})\, e^{-\left(\frac{k}{c_v}+\alpha\right)t}$$

or

$$T - T_1 = (T_0 - T_1)\, e^{-\left(\frac{k}{c_v}+\alpha\right)t}, \quad \cdots \quad (31)$$

and

$$p = p_0 \frac{T}{T_0} \left\{ e^{-(b\gamma+\alpha)t} - e^{-\left(b\gamma+\frac{k}{c_v}\right)t} \right\} + p_0 e^{-\left(b\gamma+\frac{k}{c_v}\right)t} \quad \cdots \quad (32)$$

The time t_1, at which the temperature of the gas becomes equal to the temperature of the boundary, is obtained frome (29′) by putting $T = T_1 = T_{10}e^{-\alpha t}$ in it.

$$1 = \frac{k}{k+R'b-c_v\alpha} \left\{ 1 - e^{\left(\alpha - \frac{k+R'b}{c_v}\right)t_1} \right\} + \frac{T_0}{T_{10}} e^{\left(\alpha - \frac{k+R'b}{c_v}\right)t_1} \quad \cdots\cdots \quad (33)$$

$$= \frac{k}{k+R'b-c_v\alpha} + \left(\frac{T_0}{T_{10}} - \frac{k}{k+R'b-c_v\alpha} \right) e^{\left(\alpha - \frac{k+R'b}{c_v}\right)t_1},$$

$$e^{\left(\frac{k+R'b}{c_v}-\alpha\right)t_1} = \frac{\triangle T_0 k + T_0(R'b-c_v\alpha)}{T_{10}(R'b-c_v\alpha)} = \frac{T_0}{T_{10}} + \frac{\triangle T_0}{T_{10}} \frac{k}{R'b-c_v\alpha}, \quad \triangle T_0 = T_0 - T_{10}$$

or

$$t_1 = \frac{c_v}{k+R'b-c_v\alpha} \log\left(\frac{T_0}{T_{10}} + \frac{\triangle T_0}{T_{10}} \frac{k}{R'b-c_v\alpha} \right) \quad \cdots\cdots \quad (34)$$

or

$$t_1 = \frac{c_v}{k+\delta} \log\left(1 + \frac{\triangle T_0}{T_{10}} + \frac{\triangle T_0}{T_{10}} \frac{k}{\delta} \right), \quad \delta = R'b - c_v\alpha. \quad \cdots\cdots \quad (34')$$

If the values of $\frac{\triangle T_0}{T_{10}}$ and $\frac{k}{\delta}$ is very small compared to 1, (34′) becomes

$$t_1 = \frac{c_v}{\delta} \left(1 - \frac{k}{\delta} \right) \frac{\triangle T_0}{T_{10}}. \quad \cdots\cdots \quad (34'')$$

Thus t_1 decreases from the value $\frac{c_v}{\delta} \frac{\triangle T_0}{T_{10}}$ as the value of k increases from 0.

(ii) The case where the pressure of the gas slowly changes according to $p = p_0 e^{-\alpha t}$ by the change of the boundary.

In this case the energy equation is (25), and the solution is obtained by similar way with the case 4. (ii), considering the initial condition that at $t = 0$, $T = T_0$. Thus the solution of (25) is expressed by

$$T = \frac{kT_{10}}{k+R'c-c_p\alpha} e^{-\alpha t} + \left(T_0 - \frac{kT_{10}}{k+R'c-c_p\alpha}\right) e^{-\frac{k+R'c}{c_p}t} \quad \cdots\cdots\cdots (35)$$

or

$$T = \frac{kT_{10}}{k+R'c-c_p\alpha}\left(e^{-\alpha t} - e^{-\frac{k+R'c}{c_p}t}\right) + T_0 e^{-\frac{k+R'c}{c_p}t} \quad \cdots\cdots\cdots\cdots (35')$$

In a special case where $\frac{T_0}{T_{10}} = \frac{k}{k+R'c-c_p\alpha}$ is satisfied,

$$T = T_0 e^{-\alpha t}.$$

The time variation of volume is obtained as follows.

$$v = \frac{R'T}{p} = \frac{R'kT_{10}}{p_0(k+R'c-c_p\alpha)}\left\{e^{(c-\alpha)t} - e^{\left(c-\frac{k+R'c}{c_p}\right)t}\right\} + \frac{R'T_0}{p_0} e^{\left(c-\frac{k+R'c}{c_p}\right)t}$$

or

$$v = v_0 \frac{kT_{10}}{T_0(k+R'c-c_p\alpha)}\left\{e^{(c-\alpha)t} - e^{\left(\frac{c}{\gamma}-\frac{k}{c_p}\right)t}\right\} + v_0 e^{\left(\frac{c}{\gamma}-\frac{k}{c_p}\right)t} \quad \cdots\cdots (36)$$

If $\alpha = \frac{R'c}{c_p}$ is satisfied, (35) and (36) become

$$T - T_1 = (T_0 - T_{10}) e^{-\left(\frac{k}{c_p}+\alpha\right)t} \quad \cdots\cdots\cdots\cdots\cdots\cdots (37)$$

and

$$v = v_0 \frac{T_{10}}{T_0}\left\{e^{\frac{c}{\gamma}t} - e^{\left(\frac{c}{\gamma}-\frac{k}{c_p}\right)t}\right\} + v_0 e^{\left(\frac{c}{\gamma}-\frac{k}{c_p}\right)t} \quad \cdots\cdots\cdots\cdots (38)$$

respectively.

To obtain the value of t_1, at which T becomes equal to T_1, we consider the ratio $\frac{T}{T_1}$. Dividing (35') with $T_1 = T_{10} e^{-\alpha t}$, we get

$$\frac{T}{T_1} = \frac{k}{k+R'c-c_p\alpha}\left\{1 - e^{\left(\alpha-\frac{k+R'c}{c_p}\right)t}\right\} + \frac{T_0}{T_{10}} e^{\left(\alpha-\frac{k+R'c}{c_p}\right)t} \quad \cdots\cdots (39)$$

From this t_1 is obtained as follows.

$$\frac{R'c-c_p\alpha}{k+R'c-c_p\alpha}=\left(\frac{T_0}{T_{10}}-\frac{k}{k+R'c-c_p\alpha}\right)e^{\left(\alpha-\frac{k+R'c}{c_p}\right)t_1}$$

$$t_1=\frac{c_p}{k+R'c-c_p\alpha}\log\left(\frac{T_0}{T_{10}}+\frac{\triangle T_0\, k}{T_{10}(R'c-c_p\alpha)}\right),\quad \triangle T_0=T_0-T_{10},\cdots \quad (40)$$

or

$$t_1=\frac{c_p}{k+\varepsilon}\log\left(1+\frac{\triangle T_0}{T_{10}}+\frac{\triangle T_0}{T_{10}}\frac{k}{\varepsilon}\right),\quad \varepsilon=R'c-c_p\alpha. \cdots\cdots \quad (40')$$

If the values of $\frac{\triangle T_0}{T_{10}}$ and $\frac{k}{\varepsilon}$ is very small compared to 1, (40′) becomes

$$t_1=\frac{c_p}{\varepsilon}\left(1-\frac{k}{\varepsilon}\right)\frac{\triangle T_0}{T_{10}}. \cdots\cdots \quad (40'')$$

Thus t_1 decreases from $\frac{c_p}{\varepsilon}\frac{\triangle T_0}{T_{10}}$ as the value of k increases from 0.

6. Conclusions

As a simple example of the variations of the general system, produced by the slow changes of the boundary, we studied various changes of a gas system, under the influence of the variations of the boundary condition. Here we have studied the case where the temperature of the boundary is constant, and the case where the temperature of the boundary varies exponentially with time. In both cases, the variation of pressure or volume of the gas system, produced by the variation of the boundary, is given, and it is assumed that heat transfers from the boundary to the gas in proportion to the difference of temperatures of the boundary and the gas. We have obtained the time variations of temperature and volume or pressure of the gas system. We shall study the changes of the other general systems on another occasion.

In conclusion, the writer wishes to express sincere thanks to Prof. S. T. Nakamura for his guidance and valuable suggestions.

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(1) S. Fujita: Quasi-Stationary Process (I).
Kumamoto J. Sc. vol. 2. No. 1. (1955) 51.
(2) S. Fujita: Quasi-Stationary Process (II).
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ON THE SURFACE FIGURE OF THE EARTH, THE MOON AND MARS. (Report V).

Saemon Taro NAKAMURA

(Received November 30, 1955)

Chapter III. The general expression in the Harmonic series for the mars.

The coefficients of the general harmonic series for the mars were computed by using the data given in Table 7. The results are given in Table 9.

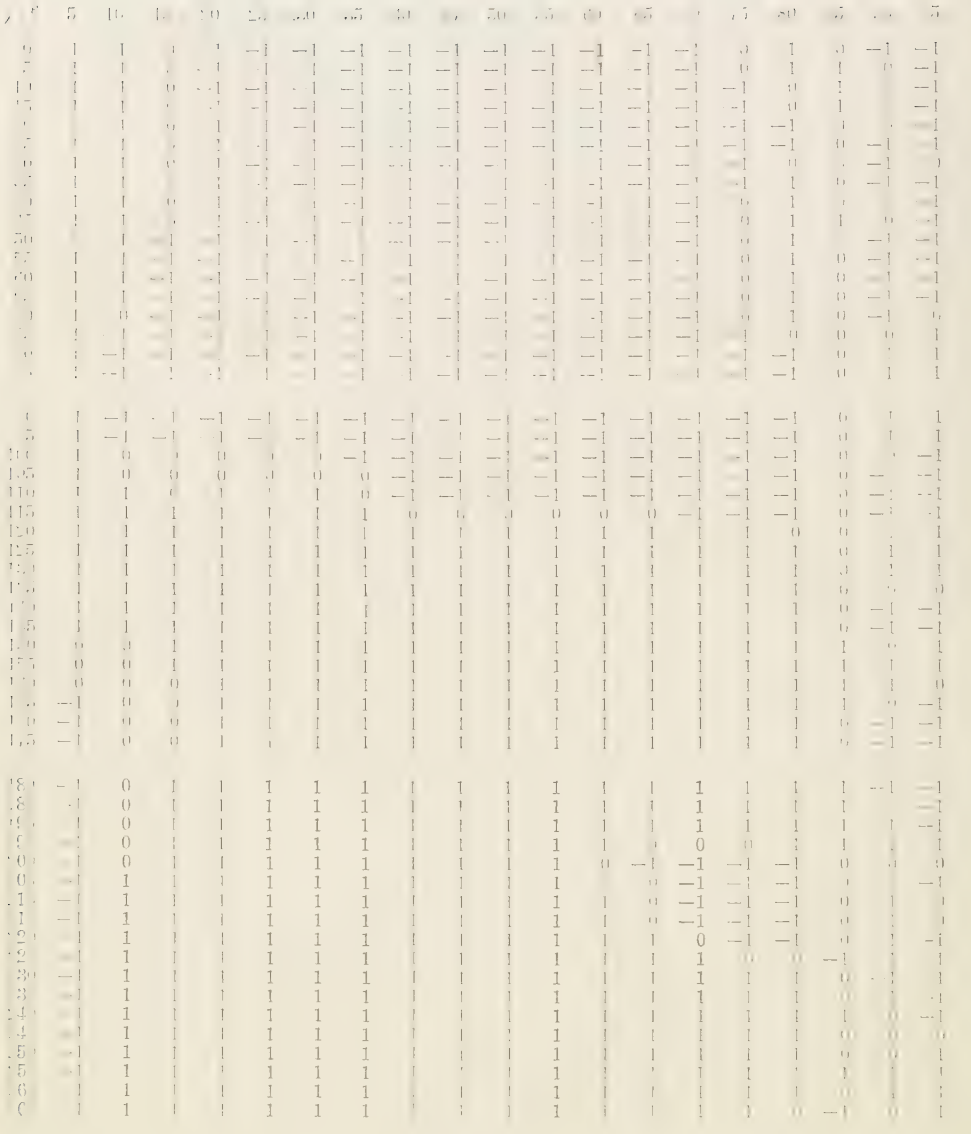
Table 9.

m \ n	0	1	2	3	4	5
0	0.1798	0.6507	0.01164	−0.5510	0.1798	0.2186
1 A		−0.2638	0.06245	0.04295	-6.097×10^{-2}	0.01347
1 B		0.3581	0.08967	−0.1229	5.425×10^{-2}	0.03141
1 $\sqrt{A^2+B^2}$		0.4448	0.1093	0.1301	8.161×10^{-2}	0.03418
2 A			0.04657	0.5800×10^{-2}	-5.271×10^{-3}	-0.081×10^{-2}
2 B			−0.09326	-1.882×10^{-2}	0.215×10^{-3}	-1.036×10^{-2}
2 $\sqrt{A^2+B^2}$			0.1042	1.970×10^{-2}	5.275×10^{-3}	1.039×10^{-2}
3 A				-1.437×10^{-3}	0.599×10^{-3}	1.323×10^{-3}
3 B				-1.120×10^{-3}	-2.657×10^{-3}	-0.026×10^{-3}
3 $\sqrt{A^2+B^2}$				1.822×10^{-3}	2.723×10^{-3}	1.323×10^{-3}
4 A					7.742×10^{-4}	-7.496×10^{-6}
4 B					3.371×10^{-4}	-0.525×10^{-6}
4 $\sqrt{A^2+B^2}$					8.444×10^{-4}	7.514×10^{-6}
5 A						4.183×10^{-5}
5 B						0.400×10^{-5}
5 $\sqrt{A^2+B^2}$						4.202×10^{-5}

Chapter IV. The General Expression for the Moon.

In the case of the moon, about 59% of its surface is visible. It is known that in the case of the earth the expression of the surface figure computed using the data for the old continents or between 0° and 210° E in longitudes (about 58% of the whole earth) can be roughly extrapolated to the remaining parts, as seen in chapter I. It may be probable that the same conclusion can be also applied to the case of the moon, as the three bodies (the earth, the moon and mars) are in quite similar state. If this assumption be true, something of the backside of the moon may be obtained by determining the coefficients of the harmonic series using the data of the visible part. To simplify the computation, the coordinates are transformed at first. The new polar axis is taken in the direction of the mean centre of the moon's disk. The coordinates are the polar distance (θ) from this new axis and the longitude (λ) counted eastwards from the north pole of the moon. Table 10 gives the data of elevation referred to these new coordinates.

Table 10.



λ \ θ	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95
270	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	−1	−1	1
275	−1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	−1	−1	−1	−1
280	−1	1	1	1	1	1	1	1	1	1	1	1	1	0	−1	−1	−1	−1	0
285	−1	0	1	1	1	1	1	1	1	1	1	0	0	−1	−1	−1	−1	−1	1
290	−1	0	1	1	1	1	1	1	1	1	0	0	−1	−1	−1	−1	−1	1	1
295	−1	0	1	1	1	0	0	0	0	0	−1	−1	−1	−1	−1	−1	0	−1	0
300	−1	−1	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1
305	−1	−1	0	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1
310	−1	−1	0	0	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1
315	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1
320	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1
325	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	0	0	0	−1	−1	−1
330	0	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1	1	1	1	1	−1	−1	−1
335	0	0	−1	−1	−1	−1	−1	−1	−1	0	1	1	1	1	1	1	0	−1	0
340	0	0	−1	−1	−1	−1	−1	−1	−1	0	1	1	1	1	1	1	1	1	1
345	0	0	0	−1	−1	−1	−1	−1	−1	0	0	1	1	1	1	1	1	0	−1
350	1	1	0	−1	−1	−1	−1	−1	−1	−1	−1	0	0	0	1	1	1	1	−1
355	1	1	0	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	1	1	0	−1

The computations carried out in two ways to obtain the coefficients. In the first, terms up to the third order and in the second up to the fifth order are taken. The results are given in Tadle 11.

Table 11a.

m \ n		0	1	2	3
0	A	−0.9848	2.291	−1.901	0.8577
1	A		−0.4757	−0.2800	−0.2634
	B		−0.2694	−0.0908	−0.1529
2	A			0.01805	0.02617
	B			−0.05443	0.00026
3	A				0.01283
	B				0.01408

Table 11b

m	n	0	1	2	3	4	5
0	A	−10.74	25.95	−27.27	18.35	− 7.772	1.766
1	A		− 2.377	− 1.833	− 1.806	− 0.7486	− 0.1961
	B		− 4.676	4.749	− 3.609	1.613	− 0.3624
2	A			− 0.8935	0.7170	− 0.3612	0.09546
	B			− 0.3234	0.2068	− 0.1302	0.03757
3	A				-6.594×10^{-3}	0.01288	1.941×10^{-3}
	B				3.718×10^{-3}	0.00809	1.332×10^{-3}
4	A					0.358×10^{-3}	-0.096×10^{-3}
	B					-6.942×10^{-3}	1.906×10^{-3}
5	A						-0.89×10^{-4}
	B						-2.14×10^{-4}

The results of the calculations are graphically shown in Fig. 4 and 5. In the both of cases, the backside of the moon is generally depressed.

It cannot be decided at present whether the present result represents the true idea of the surface figure of the backside of the moon. But the depression in the backside seems to

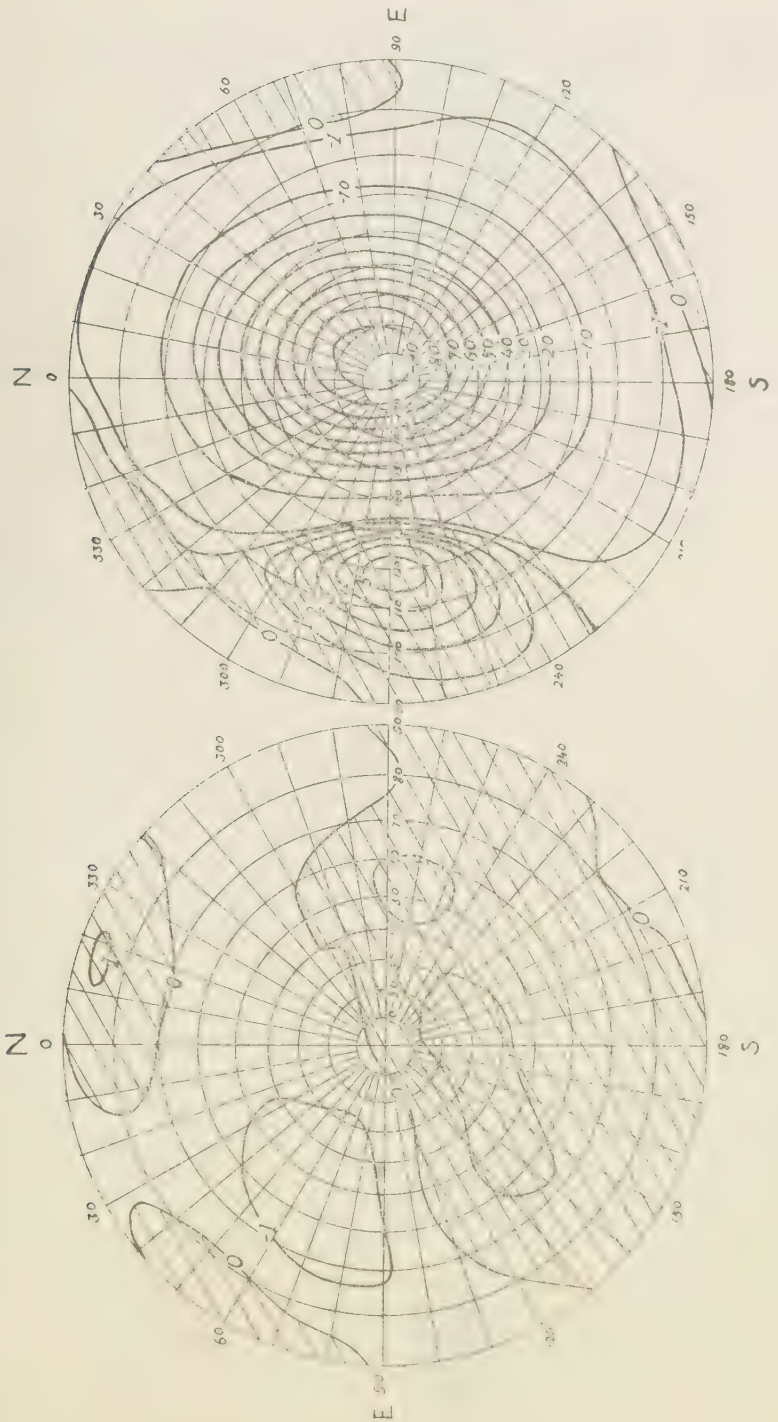
be probable as the two independent calculations indicate the same tendensy. It is also probable that such distribution of mass on the moon can be possible one from the point of view of the stability of the moon, in the gravitational field of the earth.

Conclusion

Lamb's expression of the surface figure of the earth has no physical meaning, as it is possible that any one valued continuous function on a sphere can be expressed by harmonic function. But that the function determined by using the data on a part only can be approximately extraporated to the remaining part as the present author shows, indirectly proves the existence of some simple physical meaning of the present figure of the earth.

The author found that the coefficients of the zonal terms for the earth, and mars have some common tendency, and an opposite tendency in the earth and the moon. The result can be explained by assuming some common agency which stimulates the oscillations of these three bodies.

The backside of the moon was calculated as in the case of the earth by extraporated the figure facing to the earth. The result shows that the backside of the moon is depressed, or the moon is pearshaped, which seems to be a possible stable form in the view of stability of its present position in the gravitational field of the earth.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-70
-80
-90

Fig. 4

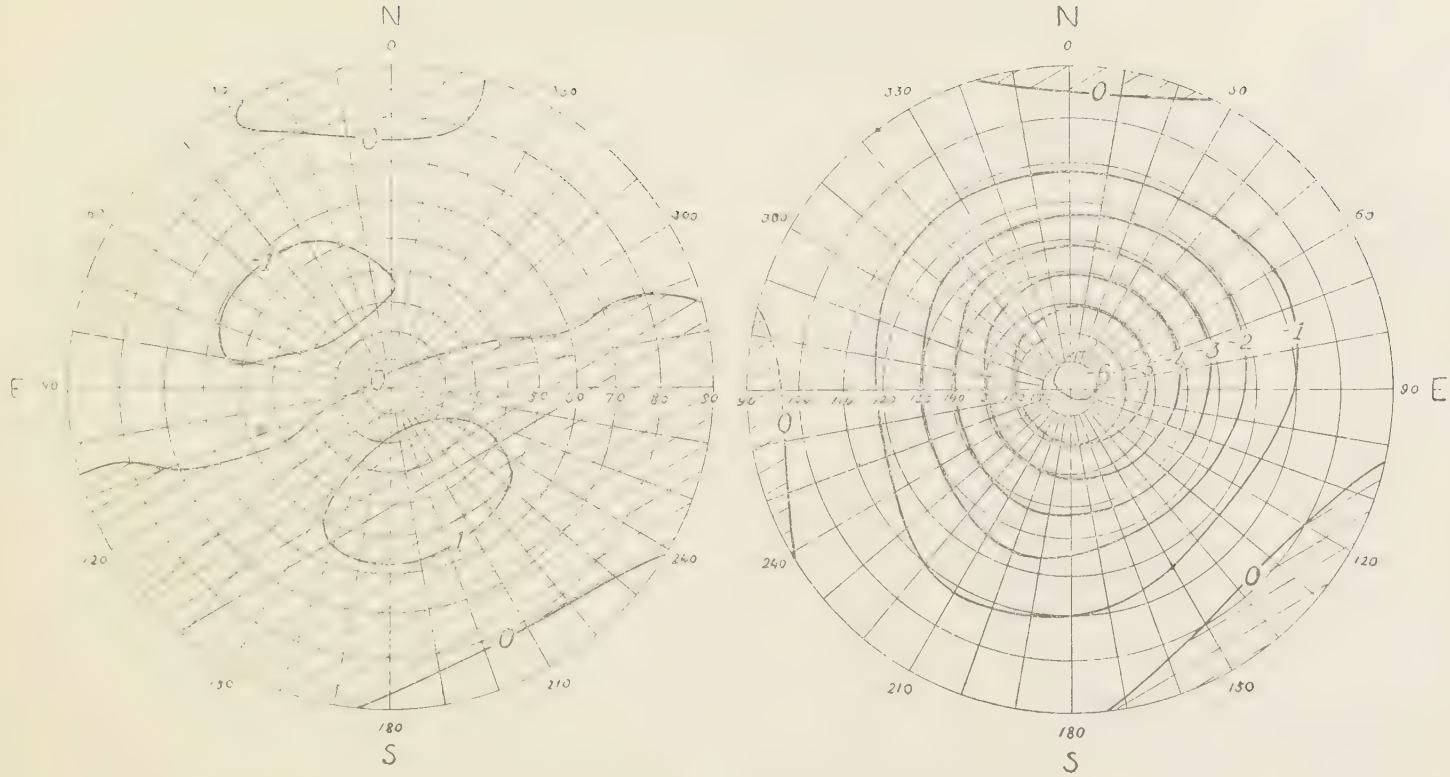


Fig. 5

THE EVOLUTION OF PARTIALLY DEGENERATE STANDARD STELLAR MODELS

Keisuke KAMINISI

(Received December 5, 1955)

Abstract

The luminosities of partially degenerate standard stellar models are calculated by making use of the results given by G. W. Wares under the assumption of radiative equilibrium in the outer layer. The present resultes make it possible to draw the evolutionary paths in the Hertzsprung-Russell diagram and to estimate the lifetimes of these models. The present work suggests that white dwarfs have, in a *short* period of time, come from the main-sequence stars passing through the *earlier* spectral type region in the H-R diagram.

1. Introduction

At present it is considered that white dwarfs arise through the gravitational collapse of fuel-exhausted masses below $5.75\ M_\odot/\mu_e^2$ (Chandrasekhar's limit) produced in the interior of massive stars [1]. In order to investigate the evolutionary paths of such burnt-out materials in the H-R diagram, from the onset of degeneracy at their centre to the appearance of degeneracy throughout most of them, we make use of the results given by G. W. Wares for the partially degenerate standard stellar models [2].

We calculate the luminosities of such models of given masses for three stages of degeneracy at their centre ($\Psi_0 = 0, 2$ and 5) under the assumptions of radiative equilibrium in the outer layer and of Kramers' law for the opacity. From these results we estimate the time interval between the stages of degeneracy.

2. Luminosity

The luminosity of a star L(R) is the total amount of energy flow per unit time passing through the surface of a star (radius R). On the surface of a star, however, some of the physical quantities have unfavourable singularities, then we calculate the energy flux L(r) passing through the sphere of radius r, which is nearly as large as the radius of the star, and replace L(R) by L(r). In other words we assume the relation: $L(R) \simeq L(r)$, where $R - r \ll R$. This approximation may be considerably accurate, because in such a thin outer fringe of a star there exists no appreciable energy source. Practically speaking we estimate the luminosity at the place of $\Psi = -4.0$ where Ψ indicates the degree of degeneracy. $\Psi = -4.0$ is equivalent to $A = 0.0183 \ll 1$, where A is Sommerfeld's criterion of degeneracy. Therefore, in the region we are considering, the material is considered to be in a nearly non-degenerate state. In this region conductive transfer of energy is negligible compared with radiation transfer.

Then we can get the following relation: *

$$L(R) \simeq L(r) = -4\pi r^2 \frac{c}{\kappa\rho} \frac{dp_r}{dr},$$

$$= -\frac{4\pi r^2 c}{\kappa\rho}(1-\beta)\frac{dP}{dr},$$

$$= -\frac{4\pi cG(1-\beta)}{\kappa} M(r), \quad \cdots\cdots (1)$$

where p_r = radiation pressure, κ = absorption coefficient,

G = gravitational constant, $M(r) = 4\pi \int_0^r r^2 \rho dr \simeq M(R)$,

$(1-\beta) = \frac{p_r}{P}$; a constant by the definition of standard model.

Now we adopt Kramers' law as the absorption coefficient:

$$\kappa = \frac{\kappa_0 \rho}{tT^{3.5}} \quad \text{with } \kappa_0 = 3.9 \times 10^{25}(1 - X^2).$$

Here t is the guillotine factor and its value is assumed to be 5 under the circumstances in question. X is the hydrogen content and its value is adopted to be 0.7 due to the high hydrogen content in the outer layer.

In our partially degenerate standard model [2]

$$\rho = 2.76\times10^6 \mu_e \frac{1-\beta}{\beta^3} \frac{2}{3} F_{3/2}(\Psi) F_{1/2}(\Psi)\ gm/cm^3,$$

$$T = 4.47\times10^9 \left\{\frac{2}{3} F_{3/2}(\Psi)\right\}^{2/3} \left(\frac{1-\beta}{\beta}\right)^{2/3} {}^\circ K,$$

where both $F_{1/2}(\Psi)$ and $F_{3/2}(\Psi)$ are the Fermi-Dirac functions defined by

$F_v(\Psi) = \int_0^\infty \frac{u^v du}{e^{-\Psi+u}+1}$, hence $F_{1/2}(-4.0)$ and $F_{3/2}(-4.0)$ are read from the available table [3].

---

* In our model $\left(\frac{dT}{T}\right)_{ad} = \frac{2(4-3\beta)}{32-24\beta-3\beta^2} \frac{dP}{P} \simeq \frac{2}{5} \frac{dP}{P}$ $(\because \beta \simeq 1)$

and $\left(\frac{dT}{T}\right)_{rad} = \frac{1}{4} \frac{dp_r}{p_r} = \frac{1}{4} \frac{dP}{P}$; that is $\left(\frac{dT}{T}\right)_{ad} > \left(\frac{dT}{T}\right)_{rad}$,

this inequality indicates that the radiative gradient is stable and convective transfer of energy cannot occur.

Substituting all the above results into Eq. (1), we get

$$L = 6.932 \times 10^{33} \frac{\mathfrak{M}}{\mu_e^3} \frac{(1-\beta)^{7/3}}{\beta^{4/3}} \ erg/sec.,$$

where $\mathfrak{M} = \frac{M\mu_e}{M_\odot}$.

As for the value of μ_e, we may simply assume $\mu_e = 2$, due to the following reasons;

(i) the fuel-exhausted star consists of abundant helium in its bulk, and

(ii) in the envelope where we are considering the energy flow, high hydrogen content results to decrease the value of μ_e, but the effect of heavy nuclei capturing the electrons around them functions to increase the value of μ_e. (Almost all hydrogen atoms are thermally ionized in this region.)

Hence we get the following relation:

$$\frac{L}{L_\odot} = 2.329 \times 10^5 \mathfrak{M} \frac{(1-\beta)^{7/3}}{\beta^{4/3}} . \qquad \cdots\cdots (2)$$

This relation may be called "*Mass-Degeneracy-Luminocity Relation* of partially degenerate standard stellar model", for β is determined by the given values of $\mathfrak{M}$ and Ψ_0 (the degree of degeneracy at the centre). The values of β are culculated by the following "generalized quartic equation" [2]:

$$\frac{\mathfrak{M}}{J_M(\Psi_0)} = \left(\frac{1-\beta}{\beta^4}\right)^{1/2}, \qquad \cdots\cdots (3)$$

where

$$J_M\ (0) = 21.08,$$

$$J_M\ (2) = 31.2,$$

and $$J_M\ (5) = 71.8. \qquad **$$

Thus obtained values of $(1-\beta)$ were shown in the second column in table (1). Next using these results, we calculated the values of $\frac{L}{L_\odot}$ by Eq. (2) and showed them in the third column in table (1). ***

** In the case of $(1-\beta) \ll 1$ we can, by using Eq. (3), rewrite the "Mass-Degeneracy-Luminosity Relation" as follows:

$$\frac{L}{L_\odot} = 2.329 \times 10^5\ \mathfrak{M}^{5\frac{2}{3}} J_M^{-4\frac{2}{3}}\ (\Psi_0).$$

*** All the results given by G. W. Wares were calculated without considering the relativi-

(Continued on the next page)

Table (1)

Physical Characteristics of Models

$\mathfrak{M}$	Ψ_0	$1-\beta$	$\frac{L}{L_\odot}$	M_{bol}	$\mathfrak{R}$	T_c	$\log_{10} T_c$	$-\Omega / G\frac{M_\odot^2}{R_\odot}$	$-E / G\frac{M_\odot^2}{R_\odot}$
		$\times 10^{-4}$				$\times 10^3$			
0.4	0	3.602	8.604×10^{-4}	12.51	0.345 4	3.37	3.37	0.080	0.040
	2	1.640	1.373×10^{-4}	14.51	0.139 6	2.35	3.37	0.170	0.085
	5	0.3103	2.819×10^{-6}	18.73	0.077 9	1.19	3.08	0.256	0.128
1.0	0	22.51	1.551×10^{-1}	6.94	0.254 5	10.10	4.00	0.684	0.343
	2	10.25	2.471×10^{-2}	8.87	0.102 9	10.04	4.00	1.45	0.726
	5	1.939	5.070×10^{-4}	13.09	0.057 4	5.09	3.71	2.18	1.088
1.4	0	44.12	1.047	4.80	0.227 5	17.22	4.24	1.51	0.760
	2	20.08	1.663×10^{-1}	6.80	0.092 0	17.10	4.23	3.18	1.588
	5	3.801	3.414×10^{-3}	11.02	0.051 3	8.66	3.94	4.78	2.388
2.0	0	87.0	7.336	2.69	0.203 7	29.61	4.47	3.389	1.71
	2	40.4	1.220	4.63	0.082 04	29.80	4.47	7.258	3.64
	5	7.74	2.565×10^{-2}	8.83	0.045 55	15.23	4.18	10.95	5.48
3.0	0	188	6.726×10	0.28	0.179 8	54.83	4.74	8.636	4.40
	2	89.2	1.168×10	2.18	0.072 01	55.93	4.75	18.59	9.38
	5	17.3	2.526×10^{-1}	6.34	0.039 84	28.84	4.46	28.15	14.10
4.0	0	317	3.093×10^{2}	− 1.38	0.165 5	83.69	4.92	16.70	8.61
	2	155	5.655×10	+ 0.47	0.065 86	86.77	4.94	36.16	18.4
	5	30.7	1.277	+ 4.11	0.036 24	45.33	4.66	55.04	27.6

3. Positions of Models in the H-R Diagram

The absolute magnitude M_{bol} and the effective temperature T_e of our model are calculated by the following relations respectively:

$$M_{bol} = 4.85 - 2.5 \log \frac{L}{L_\odot},$$

and

$$\frac{T_c}{T_\odot} = \left(\frac{L}{L_\odot}\right)^{1} \left(\frac{\mu_e}{\mathfrak{R}}\right)^{1} = \left(\frac{L}{L_\odot}\right)^{1/4} \left(\frac{2}{\mathfrak{R}}\right)^{1/2} \quad \text{with } \mu_e = 2.$$

In the last relation the values of $\mathfrak{R} = \frac{R\mu_e}{R_\odot} = J_R(\Psi_0)\,[\beta^2(1-\beta)]^{-\frac{1}{6}}$ are determined by the given $\mathfrak{M}$ and Ψ_0 [2],

where $J_R(0) = 0.0921,$

$J_R(2) = 0.0827,$

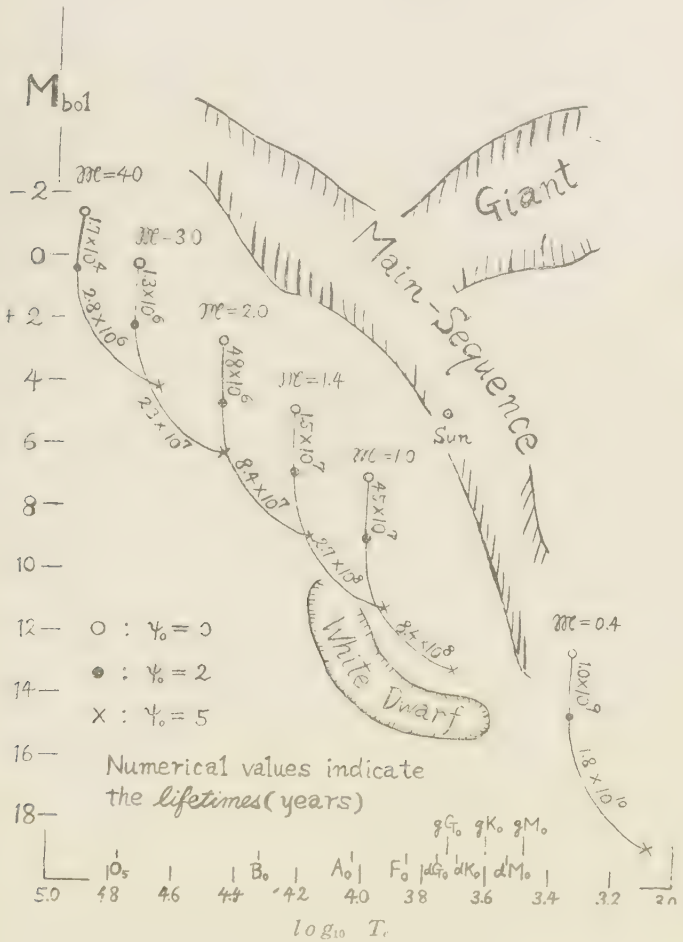
and $J_R(5) = 0.01380.$

ty effect, hence we cannot use his results for the large values of $\mathfrak{M}$ and Ψ_0. In the most dengerous case, $\mathfrak{M} = 4$ and $\Psi_0 = 5$, we have $\rho_c = 2.37 \times 10^6\ gm/cm^3$ (density at the centre), which is about the critical value of the setting-in of the relativity correction. However, since the relativity effect goes rapidly away as we proceed apart from the centre of a star, we may consider negligible the effect in the general situation.

These calculated values were shown in the fourth, fifth, sixth and seventh columns in table (1).

Thus using the above obtained results we can plot the positions of our models in the H-R diagram for the given values of $\mathfrak{M}$ and Ψ_0. These results were illustrated in the figure.

The H-R Diagram



4. Lifetime

G. W. Wares applied the partially degenerate standard stellar models to the subdwarfs and the old novae in the case of small masses [2]. Now regarding these models as the stars in the collapsing stages after burning out of their nuclear fuel, we will calculate the energy of the model at every stage of $\Psi_0 = 0$, 2 and 5, and will roughly estimate the lifetime of these models.

Although the total energy of the burnt-out stellar material consists of gravitational, electrostatic, radiative and thermal energy, we need not consider electrostatic energy and kinetic energy of heavy particles as G. W. Wares did. Then we can get the following internal energy:

$$U = \int dU = 4\pi \int_0^R r^2 \left[4\pi kT \frac{(2mkT)^{3/2}}{h^3} \int_0^\infty \frac{x^{3/2} dx}{e^{x-\Psi}+1} + aT^4 \right] dr,$$

$$= 4\pi \int_0^R r^2 \left[\frac{3}{2} p_g + 3p_r \right] dr,$$

$$= 3\left(1 - \frac{\beta}{2}\right) \int P dv = \left(1 - \frac{\beta}{2}\right) \Omega,$$

where Ω is the negative potential energy of a star.
Hence the total energy of the star E is

$$E = U + (-\Omega) = -\frac{\beta}{2}\Omega. \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

Ω is given by the following relation [4]:

$$\Omega = 2.51 \times 10^5 \; G \frac{M_\odot^2}{R_\odot \mu^2} \frac{(1-\beta)^{7/6}}{\beta^{11/3}} I(\Psi_0),$$

where $$I(\Psi_0) = \int_0^{\xi_1} \left[-\left\{ \frac{2}{3} F_{3/2}(\Psi) \right\}^{2/3} \xi^2 \frac{d\Psi}{d\xi} \right] \xi \frac{2}{3} F_{3/2}(\Psi) F_{1/2}(\Psi) \, d\xi$$

and
$$I(0) = 2.648,$$
$$I(2) = 14.097,$$
$$I(5) = 148.26.$$

We showed the values of $\frac{\Omega}{R_\odot} \frac{M_\odot^2}{}$ for given values of $\mathfrak{M}$ and Ψ_0, adopting $\mu = 2$, in the eighth column. Next we calculated the values of $-E/G\frac{M_\odot^2}{R_\odot}$ by Eq. (4) and showed them in the nineth column in table (1).

The lifetimes are roughly evaluated by the following calculation:

$$\int_{t_i}^{t_f} dt = -\int_{E_i}^{E_f} \frac{dE}{L} \simeq \frac{-(E_f - E_i)}{\sqrt{L_i L_f}}.$$

that is
$$t_f - t_i = 7.76 \times 10^6 \frac{E_i - E_f}{\sqrt{\left(\frac{L}{L_\odot}\right)_i \left(\frac{L}{L_\odot}\right)_f} \frac{M_\odot}{R_\odot} G} \text{ years.}$$

These results were tabulated in table (2) and written in the figure.

Table (2)
Lifetime of Models
(in years)

$\mathfrak{M}$	$t_2 - t_0$	$t_5 - t_2$
0.4	1.0×10^9	1.8×10^{10}
1.0	4.5×10^7	8.4×10^8
1.4	1.5×10^7	2.7×10^8
2.0	4.8×10^6	8.4×10^7
3.0	1.3×10^6	2.3×10^7
4.0	1.7×10^4	2.8×10^6

Suffix denotes the value of Ψ_0, the degree of degeneracy.

5. Conclusons

The loci of our models drawn in the H-R diagram show the evolutionary paths along which the fuel-exhausted stars are proceeding as they are settling down into the perfect degenerate states (white dwarfs). Although it is difficult to estimate the values of $\mathfrak{M}$ because of having few reliable data for the masses of white dwarfs, we may adopt, without serious mistake, the values $\mathfrak{M}=1.0 \sim \mathfrak{M}=4.0$ for our present purpose. Then by our figure we can satisfactorily illustrate the situation in which the burnt-out stars are settling down into the white dwarf stages.

Now we shall point out some features of our results:

(i) The more massive the star is, the earlier its spectral type and the brighter its luminosity for the same values of Ψ_0.

(ii) The spectral types of the stars of the smaller values of degeneracy ($\Psi_0 = 0$ and 2) are earlier than that of the stars of the larger value of degeneracy ($\Psi_0 = 5$) and the luminosities of the former are very large compared with those of the latter for the same values of $\mathfrak{M}$.

(iii) The lifetimes of our models are very short, except in the case of exceptionally small masses, compared with the galactic time scale because of their considerably large luminosities. Such a tendeney is remarkable for massive stars.

These results suggest that white dwarfs have come from the main-sequence stars *passing through the earlier spectral type region* in the H-R diagram as the author mentioned in the previous paper [4]; and the evolutionary time interval between the onset of degeneracy at the centre of the star, and the appearance of degeneracy throughout most of the star is very *short* compared with the galactic time scale as L. Mestel pointed out [5]. This may be the reason why few stars of early spectral types appear between the main-sequence and white dwarf stars region.

This paper was read on December 3, 1955, at the regular meeting of Kyushu Branch of the Physical Society of Japan.

It is a pleasure for me to thanks Dr. Z. Hitotuyanagi, Professor of Tohoku University, for his invaluable suggestions at the start of this work and the professors of the Department of Physics, Kumamoto University, for their constant encouragement.

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PORTABLE CALORIMETRIC APPARATUS APPLYING THE BOILING POINT OF THE ORGANIC SOLVENT

Daisei YAMAMOTO

(Received November 30, 1955)

The portable apparatus described below provides a handy manipulation of calorimetry under the constant temperature.

It has a added advantage of easy exchangeability in case of need.

The reaction tube A (Fig 1) is inserted into the inner side of a double walled vessel B and fixed by a cotton bandage C and a cork piece D.

The air mantel E of approximately 1cm thickness is formed between the reaction tube and inside of the vessel which contains the proper organic solvent such as ether and small pieces of glass to maintain the constant temperature at boiling point and is connected to a reflux condenser.

The vessel is immersed in a water bath which was heated to 40～50°c in case of ether, by a glass covered Ni-Cr spiral heater passed through the water, as illustrated in Fig 1.

Beckmann thermometer, stirrer etc are put into the reaction tube as a routine method.

Thus, the boiling ether bath can be used as a substitute for a thermostat and with a temperature buffer action of the air mantel, the ground condition for a portable calorimetry would be easily obtained.

Dimensions of the apparatus are not critical except the thickness of the air mantel.

Beside ether, several organic solvent (benzene alcohol etc) are applycable at their boiling point if necessary.

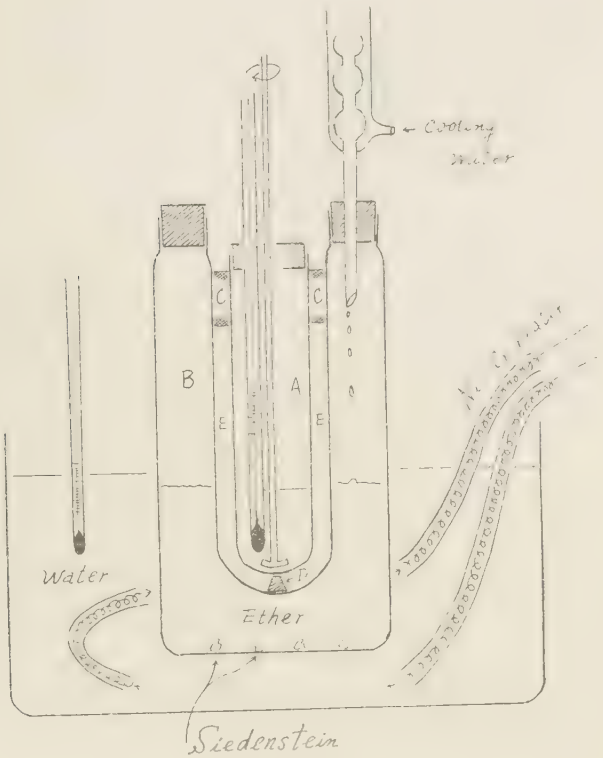

Fig 1.

Calorimetric apparatus applying the boiling point of ether

Daisei Yamamoto

22 NOV. 1955,

PAPER CHROMATOGRAPHY OF INORGANIC IONS BY USING ORGANIC ANALYTICAL REAGENTS. II. Precipitation Chromatography of Cations with 8-Quinolinol. (Part 1).

Hideo NAGAI

(Received Nov. 25, 1955)

The various chromatographic separations have mostly been done dy migrating the components of the test sample applying some sort of the driving forces and the resistive forces. For effective separations, either or both of the driving forces and the resistive forces must act selectively upon the migrating substances. The author intended to improve the separation of the substances by superposing the suppressive effect of the precipitation caused by the solubility product rule; from the analogy with the precipitation titration, the precipitation of the more soluble substances would be suppressed considerably, before the completion of the precipitation of the less soluble sobstances is attained. To utilise this effect in the chromatography, various methods may be possible, for example, the use of (1) the filter paper impregnated with the reagent which make almost insoluble salts with many ions, (2) the gelatinized reagent solution (like the experiments of the Liesegang phemomena), or (3) the ionexchanger adsorping the salt-making reagent etc. And when the test solution is developed into these reagent-supporting media, fractional precipitation chromatography may be obtained. In this paper 8-quinolinol (oxine) was used as the precipitation reagent, so the filter paper impregnated with oxine was employed.

The usual paper chromatographic studies of inorganic cations with oxine have been done by a few workers.[1),2)]

EXPERIMENTAL

Filter Paper. Toyo No. 5B. filter paper, untreated.

Solution of Cations. All reagents were of analytical grade and the test solutions were made from nitrates. Each metal salt solutions were prepared separately (each bottle contained one species of metal salt solution only, and in each experiment the solutions were mixed by the same volumes.

8-Quinolinol (oxine). Reagent grade was used without further purification.

Impregnation of Filter Paper. One hundred sheets of Toyo No. 5B. filter circle (9 cm. in diameter) were placed in a glass beaker. Then a solution of 1.2 grams of oxine in 80 ml. of 95% ethanol was added to the vessel. The paper sheets were allowed to stand for 1 hour at room temperature, then removed and shaken in order to remove most of the ethanolic solution. The paper sheets were then allowed to dry at room temperature. After drying, these treated filter paper sheets were stored in a closed vessel, and used to chromatograph

the various cations.

Spotting the Filter Paper. At the center of the oxine impregnated filter circle, a slit (1~2 mm. in length) was cut with a blade and then a drop (0.003 ~ 0.005ml.) of test solntion was spotted at the place. As soon as the solution was absorbed in the paper, a tab (1mm. width and 3 cm. length) made of untreated filter paper was inserted into the slit.

Apparatus. The apparatus used for the development of chromatograms is shown in figure 1 and is briefly described below. A large petri dish(12.5 cm. in diameter and 3 cm. in height) was inverted over a glass plate served as a chromatographic chamber. The developing solvent was kept in a small petri dish (7.5 cm. in diameter and 1.5 cm. in height) at the center of the chamber. A circular oxine-impregnated filter paper (9 cm. in diameter) was supported on the small petri dish and at the center of the filter circle a slit was cut with a blade to insert a tab made of untreated filter paper as described above.

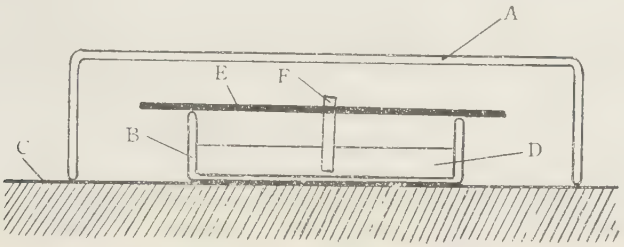

Figure 1. Apparatus of Development.

A. Large Petri dish.
B. Small Petri dish.
C. Glass plate.
D. Developing solvent.
E. Oxine-impregnated filter circle.
F. Untreated filter paper tab.

Treatment of Chromatograms. The chromatograms were allowed to develop as illustrated in figure 1. The course of development was visually observable, so the experiment was stopped as soon as the desired result was obtained. The chromatogram obtained was allowed to dry in room temperature, and then viewed under both white and ultraviolet light.

Comparison of Separability of Cations between the Usual Chromatography and the Precipitation Chromatography. To compare the separabilty of cations between the usual chromatography and the precipitation chromatography, mixed solution of Cu^{2+}, Fe^{3+}, and Co^{2+} was used as the test solution. The results obtained after development with water and various organic solvents, were as follows. The obtained pattern from the development with water was a precipitation chromatogram, and three cations were separated fairly well, but the figure was somewhat broken in some places as shown in figure 2A. This is why, the author considered, when the water (developing solvent) passed through the precipitation which had been formed, the resistance would be so large owing to the low solubility of the precipitated metal oxinates in the water that the developing solvent (water) swept off the

precipitation in some degree.

On the other hand, when the organic solvents were used (methanol, ethanol, *n*-butanol, acetone, chloroform, carbon tetrachloride and pyridine were tried) the results obtained were all inferior in the separation of cations, to that of water. In these chromatograms, developed with organic solvents, the pattern obtained with methanol showed the best result (figure 2B). In this case near the center of the circle considerable amount of Fe^{3+} and some Cu^{2+} were

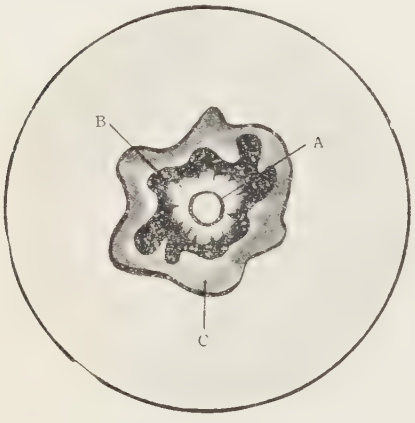

Figure 2A. Chromatogram
Developed with water.
A. Cu^{2+} zone (greenish yellow)
B. Fe^{3+} zone (black)
C. Co^{2+} zone (orange)

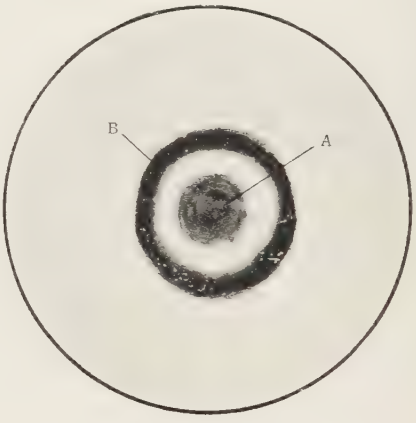

Figure 2B. Chromatogram
Developed with Methanol.
A. Mainly Fe^{3+} (dilute black)
B. Fe^{3+} and Cu^{2+} zone(dull greenish yellow)

left, and it was possible, though not clear, to discern that the inner part of the circular developed zone was abundant in Cu^{2+} and outer part of it was rich in Fe^{3+}, but Co^{2+} could not be located.

Comparing these two figures (figure 2A and 2B), precipitation chromatography is superior in separability than usual chromatography. And it may be reasonable to consider that if some organic solvent is properly added to water in order to improve the solubility, so as to be able to pass through the precipitation of metal oxinates with less resistance and the more clearly defined pattern of separate zones of cations is obtainable. Actually, when Ni^{2+} was added to the test solution used in the experiment above, the pattern obtained after developing with water alone, gave considerablely worse separations of cations (especially between Ni^{2+} and Co^{2+}) than developed with water containing some proper organic solvents.

The various developing solvents of the diluted organic solvents with water were as follows.

a) Methanol. when water was added little by little to methanol to decrease solubility of oxine salts, the chromatograms developed by the solvents changed from soluble form to precipitate form as the volume of the water increased. The most adequate ratio of methanol and water

was the range of 1 : 3~1 : 10 in volume. The order of precipitation of oxinates was Cu^{2+}, Fe^{3+}, Ni^{2+}, and Co^{2+} from the center to the circumference (this order was kept unchanged in every solvent used in this paper), but near the center of the circle was almost always retained a little amount of Fe^{3+}, accordingly the separation between Cu^{2+} and Fe^{3+} was apt to be imperfect, moreover, the separation between $Ni^{2+}$and Cu^{2+} was frequently imperfect.

b) Ethanol. Ethanol diluted with water was able to be used as the developing solvent. The dilution with water about 1 : 200 (ethanol : water, in volume) gave the best result but the good result could not be expected.

c) Acetone. Acetone properly diluted with water was able to be used as the developing solvent. The dilution with water about 2 : 5 (acetone : water, in volume) gave the best result. The chromatogram developed with this solvent was very likely resembled to that of diluted methanol.

d) *n*-Butanol. When the test solution of Cu^{2+}, Fe^{3+}, and Co^{2+} was used, the separation of cations was almost the same degree between the wide range of the dilution variety, *i. e.* from water saturated with *n*-butanol to the very dilute aqueous solution of *n*-butanol (even to the dilution of 1 : 2,000). But when the test solution of Cu^{2+}, Fe^{+3}, Co^{2+}, and Ni^{+2} was used, the pattern was somewhat broken even when the dilution of 1 : 400 *n*-butanol was used as the developing solvent. Accordingly the good result could be expected in using the more soluble solvent to the metal oxinates. In fact, water saturated with *n*-butanol was very conveniently available throughout all the year round. The result obtained with this solvent was the best one in this paper, but the separation of Cu^{2+} and Fe^{3+} was not complete but easily identified. The precipitation chromatogram developed with *n*-butanol saturated water was shown in figure 3.

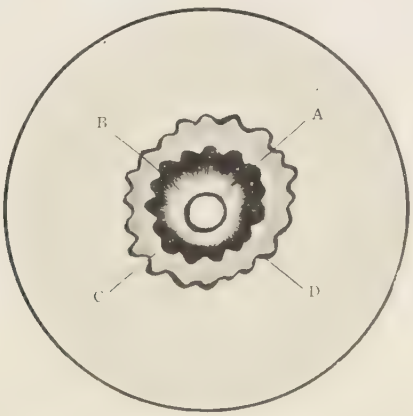

Figure 3. Chromatogram Developed with Water Saturated with *n*-Butanol.
A. Cu^{2+} zone (greenish yellow).
B. Fe^{3+} zone (black).
C. Ni^{2+} zone (light yellow-green).
D. Co^{2+} zone(orange).

CONCLUSION

1. When the metal oxinates considered in this paper was chromatographed, the precipitation chromatography, which was composed so as to precipitate the test substances from the order of the degree of insolubility by reducing the solubility of the developing solvents, was superior in separation of cations than the usual soluble form chromatography.

2. In the developing solvents tested in this paper, *n*-butanol—water solution gave the best results.

3. Comparing with the chromatography of cations which has been studied by the usual methods, the precipitation chromatography had a major point to make the visual observation of the course of development possible, so the experiment might be stopped as soon as the desired result was obtained. So the time necessary in this experiment was the sufficient and minimum.

4. In some other experiments, $CuSO_4 \cdot 5H_2O$, $FeCl_3 \cdot 6H_2O$, $CoCl_2 \cdot 6H_2O$, and $NiSO_4 \cdot 7H_2O$ were used as the test solution and the results obtained were almost the same as described in this paper using nitrates.

The content of this paper has partly been published in J. Chem. Soc. Japan, Pure Chem. Sect. **76**, 1246 (1955).

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NOTE ON BLOCKS OF GROUP CHARACTERS

Kenzo IIZUKA

(Received February 25, 1956)

By Prof. R. BRAUER a number of theorems concerning the blocks of group characters were given without detailed proofs ([2], [3], [4]). Recently, Prof. M. OSIMA determined the primitive idempotents of the center of a modular group ring and gave many properties of the idempotents. Making use of them, he proved some of Brauer's theorems ([7]).

The purpose of this paper is to verify directly some of Brauer's theorems and to derive some properties of idempotents of the center of the modular group ring. § 1 concerns with the elementary divisors of Cartan matrices corresponding to blocks and § 2 with the defect groups of blocks. In § 3 we deal with some properties of the idempotents.

Notations.

$\mathfrak{G}$: A group of finite order g.

p: A fixed rational prime number.

a: The exponent of the highest power of p dividing g.

$K_1, K_2, \cdots\cdots, K_n$: The classes of conjugate elements of $\mathfrak{G}$. K_ν^* denotes the class which consists of elements reciprocal to those of K_ν.

$\mathfrak{C}(\ )$: The centralizer of a subgroup in $\mathfrak{G}$.

$\mathfrak{N}(\ )$: The normalizer of a subgroup or an element in $\mathfrak{G}$.

n_ν: The order of the normalizer $\mathfrak{N}(G_\nu)$ of an element G_ν in K_ν.

$\mathfrak{H}_\nu$: The defect group of K_ν for p, i.e., a p-Sylow subgroup of the normalizer $\mathfrak{N}(G_\nu)$ of an element G_ν in K_ν, which is uniquely determined by the class K_ν up to conjugacy.

ρ_ν: The defect of K_ν for p, i. e., the order of $\mathfrak{H}_\nu$.

g_ν: The number of elements in K_ν, $g_\nu = g/n_\nu$.

K: The field of the g-th roots of unity.

$\mathfrak{p}$: A prime ideal of K dividing p. We also denote by $\mathfrak{p}$ the ideal $\mathfrak{p}\mathfrak{o}_\mathfrak{p}$ in the ring $\mathfrak{o}_\mathfrak{p}$ of $\mathfrak{p}$-integers of K.

K^*: The residue class field $\mathfrak{o}_\mathfrak{p}/\mathfrak{p}$.

$\Gamma(\ )$: The group ring of a subgroup of $\mathfrak{G}$ over K.

$\Lambda(\ )$: The center of $\Gamma(\ )$.

$\Gamma^*(\ )$: The group ring of a subgroup of $\mathfrak{G}$ over K^*.

$\Lambda^*(\ )$: The center of $\Gamma^*(\ )$.

$\zeta^{(1)}, \zeta^{(2)}, \cdots\cdots, \zeta^{(m)}$: The absolutely irreducible ordinary characters of $\mathfrak{G}$. $\zeta_\nu^{(i)}$ denotes

the value of $\zeta^{(i)}$ for the class K_ν.

z_i : The degree of $\zeta^{(i)}$.

$\omega^{(i)}$: The linear character of $\Lambda(\mathfrak{G})$ belonging to $\zeta^{(i)}$. $\omega_\nu^{(i)}$ denotes the value of $\omega^{(i)}$ for the class K_ν, $\omega_\nu^{(i)}=g_\nu\zeta_\nu^{(i)}/z_i$.

$\varphi^{(1)}$, $\varphi^{(2)}$, ……, $\varphi^{(k)}$: The absolutely irreducible modular characters of $\mathfrak{G}$ for p defined as complex numbers. $\varphi_\nu^{(\kappa)}$ denotes the value of $\varphi^{(\kappa)}$ for the p-regular class K_ν.

Z : The matrix $(\zeta_\nu^{(i)})_{i,\nu}$ of ordinary characters of $\mathfrak{G}$.

Φ : The matrix $(\varphi_\nu^{(\kappa)})_{\kappa,\nu}$ of modular characters of $\mathfrak{G}$.

$d_{i\kappa}$: The decomposition numbers of $\mathfrak{G}$ for p.

$c_{\kappa\lambda}$: The Cartan invariants of $\mathfrak{G}$ for p.

D : The matrix $(d_{i\kappa})_{i,\kappa}$ $(i=1, 2, \cdots\cdots, n;\ \kappa=1, 2, \cdots\cdots, k)$.

C : The matrix $(c_{\kappa\lambda})_{\kappa,\lambda}$ $(\kappa, \lambda=1, 2, \cdots\cdots, k)$.

B_1, B_2, ……, B_s: The blocks of characters of $\mathfrak{G}$ for p. Corresponding to B_1, B_2, ……, B_s if the rows and columns are arranged suitably, C and D have the forms

$$C=\begin{pmatrix} C_1 & & 0 \\ & C_2 & \ddots \\ 0 & & C_s \end{pmatrix},\quad D=\begin{pmatrix} D_1 & & 0 \\ & D_2 & \ddots \\ 0 & & D_s \end{pmatrix}.$$

From now on we assume that C and D are of these forms.

1. Let K_1, K_2, ……, K_k be the p-regular classes. Let x_τ and y_τ denote the numbers of $\zeta^{(i)}$ and $\varphi^{(\kappa)}$ in B_τ, respectively. Consider the k columns of Z which belong to K_1, K_2, ……, K_k. Then we can select a minor Δ of degree k which is not divisible by $\mathfrak{p}$. As is well known, y_τ of the rows in Δ must belong to $\zeta^{(i)}$ in B_τ. It is then possible to associate y_τ of the p-regular classes with B_τ in such a way, that each p-regular class is associated with one and only one block and that the y_τ rows of Δ belonging to $\zeta^{(i)}$ in B_τ and the y_τ columns belonging to the p-regular classes associated with B_τ form a minor which is not divisible by $\mathfrak{p}$.

〔1.1〕 *If the p-regular classes K_λ, K_μ, ……, K_σ are associated with B_τ, then the elementary divisors of C_τ are p^{ρ_λ}, p^{ρ_μ}, ……, p^{ρ_σ}.*

PROOF. Corresponding to the form of C we write

$$\Phi=\begin{pmatrix} \Phi_1 & & * \\ & \Phi_2 & \ddots \\ * & & \Phi_s \end{pmatrix}$$

where the columns containing Φ_τ correspond to y_τ p-regular classes associated with B_τ.

Then we see that

$$|\Phi_\tau| \not\equiv 0 \pmod{\mathfrak{p}} \qquad (\tau = 1, 2, \cdots\cdots, s).$$

Similarly we write

$$\Psi = (\zeta_\mu^{(\kappa)})^{-1} = \begin{pmatrix} \Psi_1 & & & * \\ & \Psi_2 & & \\ & & \ddots & \\ * & & & \Psi_s \end{pmatrix}.$$

Since $|\Phi| \not\equiv 0 \pmod{\mathfrak{p}}$, all coefficients of Ψ are $\mathfrak{p}$-integers. From the orthogonal relations for modular characters of $\mathfrak{G}$ we have

$$C_\tau \Phi_\tau = \Psi'_\tau \begin{pmatrix} n_\lambda & & 0 \\ & n_\mu & \\ & \ddots & \\ 0 & & n_\sigma \end{pmatrix}$$

where Ψ'_τ is the transpose of Ψ_τ. Therefore

$$|C_\tau| \geqq p^{\rho_\lambda + \rho_\mu + \cdots\cdots + \rho_\sigma} \qquad (\tau = 1, 2, \cdots\cdots, s).$$

On the other hand

$$|C| = \prod_\tau |C_\tau| = p^{\rho_1 + \rho_2 + \cdots\cdots + \rho_k}.$$

Hence

$$|C_\tau| = p^{\rho_\lambda + \rho_\mu + \cdots\cdots + \rho_\sigma}.$$

Thus we have

$$|\Psi_\tau| \not\equiv 0 \pmod{\mathfrak{p}},$$

and hence the elementary divisors of C_τ are p^{ρ_λ}, p^{ρ_μ}, $\cdots\cdots$, p^{ρ_σ}.

〔1.2〕 *If B_τ is a block of defect d, then the corresponding Cartan matrix C_τ has one elementary divisor p^d while all other elementary divisors are powers of p with exponents smaller than d.*

PROOF.[1] Let p^{ε_1}, p^{ε_2}, $\cdots\cdots$, p^{ε_y} $(\varepsilon_1 \leqq \varepsilon_2 \leqq \cdots\cdots \leqq \varepsilon_y,\ y = y_\tau)$ be the elementary divisors of C_τ and consider

$$S_{ij} = \sum_{\nu=1}^{k} g_\nu \zeta_\nu^{(i)} \zeta_{\nu^{-1}}^{(j)} \qquad (\zeta^{(i)}, \zeta^{(j)} \text{ in } B_\tau).$$

Then we see that

$$S_\tau = (S_{ij})_{i,j} = D_\tau g C_\tau^{-1} D'_\tau.$$

All coefficients of S_τ are rational integers and the elementary divisors of S_τ are $p^{d-\varepsilon_y}, \cdots,$

[1] This proof is due to [1]. Cf. [1], footnote 18 and § 8.

p^{a-e_2}, p^{a-e_1}, 0, ……, 0. Since every $\omega_\nu^{(i)}$ is an algebraic integer and the degree z_i of each $\zeta^{(i)}$ in B_τ is divisible at least by p^{a-d}, we have

$$S_{ij} = S_{ji} \equiv 0 \pmod{p^{a-d}}. \tag{1.1}$$

Hence we have $e_y \leqq d$. On the other hand, e_y is not less than d.[2] Consequently we have

$$e_y = d.$$

Let $\zeta^{(i_1)}$ be a character in B_τ such that z_{i_1} is exactly divisible by p^{a-d}. Since the congruences

$$\omega_\nu^{(i)} \equiv \omega_\nu^{(j)} \pmod{\mathfrak{p}} \qquad (\zeta^{(i)}, \zeta^{(j)} \text{ in } B_\tau) \tag{1.2}$$

hold for all classes K_ν of $\mathfrak{G}$, we have

$$S_{ij} \equiv \frac{z_i}{z_{i_1}} S_{i_1 j} \pmod{p^{a-d+1}}. \tag{1.3}$$

In particular

$$S_{i_1 j} = S_{j i_1} \equiv \frac{z_j}{z_{i_1}} S_{i_1 i_1} \pmod{p^{a-d+1}},$$

and therefore

$$S_{ij} \equiv \frac{z_i z_j}{z_{i_1}^2} S_{i_1 i_1} \pmod{p^{a-d+1}}.$$

Thus we have

$$S_\tau = S_{i_1 i_1}\left(\frac{z_i z_j}{z_{i_1}^2}\right) + H_\tau.$$

H_τ is a matrix in which all coefficients are divisible by p^{a-d+1}. Since every minor of degree 2 of S_τ is divisible by $p^{(a-d)+(a-d+1)}$, we have

$$e_{y-1} < d.$$

[1.3] *A character $\zeta^{(i)}$ belongs to a block of defect larger than or equal to a given number d, if and only if*

$$\omega_\nu^{(i)} \equiv 0 \pmod{\mathfrak{p}} \tag{1.4}$$

for all p-regular classes K_ν with $\rho_\nu < d$.

PROOF. Since the sufficiency is obvious, we prove the necessity. Suppose that $\zeta^{(i)}$ belongs to a block B_τ of defect d_τ smaller than d. By [1.1] and [1.2] we can find a p-regular class K_μ with $\rho_\mu = d_\tau$ among the p-regular classes associated with B_τ in the sense of [1.1]. Then we have by (1.4)

[2] See [5], §7.

$$\frac{g_\mu \zeta_\mu^{(j)}}{z_j} = \omega_\mu^{(j)} \equiv \omega_\mu^{(i)} \equiv 0 \pmod{\mathfrak{p}}$$

for all $\zeta^{(j)}$ in B_τ. Since g_μ is exactly divisible by p^{a-d_τ} and z_j is divisible at least by p^{a-d_τ},

$$\zeta_\mu^{(j)} \equiv 0 \pmod{\mathfrak{p}}$$

holds for all $\zeta^{(j)}$ in B_τ, which yields a contradiction. Therefore we have

$$d_\tau \geqq d.$$

〔1.4〕 *Two characters $\zeta^{(i)}$ and $\zeta^{(j)}$ belonging to blocks of defect d appear in the same block, if and only if*

$$\omega_\nu^{(i)} \equiv \omega_\nu^{(j)} \pmod{\mathfrak{p}} \tag{1.5}$$

for all p-regular classes K_ν with $\rho_\nu = d$.

PROOF. Since the sufficiency is obvious, we prove the necessity. Suppose that $\zeta^{(i)}$ and $\zeta^{(j)}$ belong to different blocks B_σ and B_τ, respectively. We now consider

$$S_{lm} = \sum_{\nu=1}^{k} g_\nu \zeta_\nu^{(l)} \zeta_\nu^{(m)}$$

for $\zeta^{(l)}$ and $\zeta^{(m)}$ belonging to either B_σ or B_τ. Arranging the rows and columns suitably we have

$$(S_{lm})_{l,m} = \begin{pmatrix} D_\sigma g C_\sigma^{-1} D_\sigma' & 0 \\ 0 & D_\tau g C_\tau^{-1} D_\tau' \end{pmatrix}.$$

Therefore by 〔1.2〕 two of the elementary divisors of the matrix (S_{lm}) are equal to p^{a-d}. On the other hand, from (1.5) we can deduce the following congruences

$$S_{lm} \equiv \frac{z_l}{z_{i_1}} S_{i_1 l} \pmod{p^{a-d+1}}$$

where z_{i_1} is exactly divisible by p^{a-d}. Thus we can conclude in the same way as in the proof of 〔1.2〕 that the elementary divisors of (S_{lm}) are larger than p^{a-d} except one, which yields a contradiction.

2. We also denote by K_ν the sum of all elements in the class K_ν. Then K_1, K_2, ……, K_n form a K-basis of $A = A(\mathfrak{G})$ and we have

$$K_\alpha K_\beta = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma \tag{2.1}$$

where $a_{\alpha\beta\gamma}$ are non-negative rational integers.

Let $\mathfrak{H}$ be any subgroup of order p^h of $\mathfrak{G}$ and consider a subgroup $\mathfrak{N}$ satisfying

$$\mathfrak{H}\mathfrak{C}(\mathfrak{H}) \subseteqq \mathfrak{N} \subseteqq \mathfrak{N}(\mathfrak{H}). \tag{2.2}$$

Denote by K_ν^0 the part of K_ν which lies in $\mathfrak{C}(\mathfrak{H})$. K_ν^0 is a sum of complete classes of $\mathfrak{N}$ or zero according as K_ν contains any element of $\mathfrak{C}(\mathfrak{H})$ or not. From (2.1) we can see that

$$K_\alpha^0 K_\beta^0 = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^0 \pmod{p}. \tag{2.3}$$

We show this. If we set $K_\nu = K_\nu^0 + K_\nu^1$, (2.1) yields

$$K_\alpha^0 K_\beta^0 + K_\alpha^0 K_\beta^1 + K_\alpha^1 K_\beta^0 + K_\alpha^1 K_\beta^1 = \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^0 + \sum_\gamma a_{\alpha\beta\gamma} K_\gamma^1.$$

Let m_G denote the coefficient of an element G in $K_\alpha^1 K_\beta^1$ and for each element R of $\mathfrak{C}(\mathfrak{H})$ consider the set $\mathfrak{P}_R$ of all elements P in K_α^1 such that $PQ = R$ for a suitable element Q in K_β^1. It is then easy to see that m_R is equal to the number of elements of $\mathfrak{P}_R$. In order to see that m_R is divisible by p, we consider the elements P', P'', $\cdots\cdots$ which are the transforms of an element P' in $\mathfrak{P}_R$ by the elements of $\tilde{\mathfrak{N}}(R) = \mathfrak{N}(R) \cap \mathfrak{N}$. Obviously all elements $P^{(i)}$ belong to $\mathfrak{P}_R$ and the number of elements $P^{(i)}$ is equal to the index $[\tilde{\mathfrak{N}}(R): \mathfrak{N}(P') \cap \tilde{\mathfrak{N}}(R)]$. This index is divisible by p, since $\mathfrak{H}$ is a normal subgroup of $\tilde{\mathfrak{N}}(R)$ while $\mathfrak{H}$ is not contained in $\mathfrak{N}(P') \cap \tilde{\mathfrak{N}}(R)$. Continuing the same process for the remaining P of $\mathfrak{P}_R$, we can finally see that m_R is equal to a sum of such indices. Hence m_R is divisible by p, from which (2.3) follows.

If $\mathfrak{H}_\alpha \not\equiv \mathfrak{H}^{g}$, then by (2.3) we have

$$a_{\alpha\beta\gamma} \equiv 0 \pmod{p} \tag{2.4}$$

for any class K_γ with $\mathfrak{H}_\gamma \supseteq \mathfrak{H}$. Therefore K_ν with $K_\nu^0 = 0$ form a K^*-basis of an ideal T^* in $A^* = A^*(\mathfrak{G})$ and K_ν^0 $(\neq 0)$ form a K^*-basis of a subring R^* of $\tilde{A}^* = A^*(\mathfrak{N})$. Thus we have

$$R^* \cong A^*/T^*. \tag{2.5}$$

Let $\tilde{K}_1$, $\tilde{K}_2$, $\cdots\cdots$ be the classes of conjugate elements of $\mathfrak{N}$. Then $\tilde{K}_\nu$ either does not contain any elements of $\mathfrak{C}(\mathfrak{H})$ or consists of elements of $\mathfrak{C}(\mathfrak{H})$. The classes which consist of elements of $\mathfrak{C}(\mathfrak{H})$ form a K^*-basis of a subring $\tilde{R}^*$ of $\tilde{A}^*$ and the others form a K^*-basis of an ideal $\tilde{T}^*$ in $\tilde{A}^*$. We see that $\tilde{A}^*$ is the direct sum

$$\tilde{A}^* = \tilde{R}^* + \tilde{T}^*. \tag{2.6}$$

[2.1] *If $\mathfrak{G}$ contains a normal subgroup $\mathfrak{H}$ of order p^d, then there exists at least one character $\zeta^{(i)}$ in each B_τ which represents the elements of $\mathfrak{H}$ by the unit matrix and all blocks of $\mathfrak{G}$ have at least the defect d.*

PROOF. We have

$$\sum_{H \text{ in } \mathfrak{H}} \sum_{i \text{ in } B_\tau} d_{i\kappa} \zeta^{(i)}(H) = \sum_{i \text{ in } B_\tau} d_{i\kappa} z_i \neq 0,^{4)}$$

[2]) "$\mathfrak{H}_\mu \supseteq \mathfrak{H}$" means "$\mathfrak{H}$ is conjugate to a subgroup of $\mathfrak{H}_\mu$".

because

$$\sum_{i \text{ in } B_\tau} d_{i\kappa}\, \zeta^{(i)}(S) = 0$$

holds for all p-singular elements S of $\mathfrak{G}$. Hence there exists at least one $\zeta^{(i)}$ in B_τ such that the 1-character of $\mathfrak{H}$ appears in the character $\zeta^{(i)}(H)$ of $\mathfrak{H}$. Then $\zeta^{(i)}$ represents all elements of $\mathfrak{H}$ by the unit matrix.

Let denote by ω_τ^* the linear character of A^* corresponding to B_τ.

(2.2) *If $\mathfrak{G}$ contains a normal subgroup $\mathfrak{H}$ of order p^h, then any linear character of A^* vanishes for all elements of T^*.*

PROOF. Let K_μ be a class belonging to T^* and $\overline{K}_\mu$ the class of conjugate elements of the group $\mathfrak{G}/\mathfrak{H}$ which contains the coset G_μ of an element G_μ in K_μ. The defect group $\overline{\mathfrak{H}}_\mu$ of $\overline{K}_\mu$ contains $\mathfrak{H}_\mu\mathfrak{H}/\mathfrak{H}$ while $\mathfrak{H}_\mu$ does not contain $\mathfrak{H}$. Hence we have $\overline{\rho}_\mu > \rho_\mu - h$, where $\overline{\rho}_\mu$ is the defect of $\overline{K}_\mu$. Therefore, if we denote by $\overline{g}_\mu$ the number of elements in $\overline{K}_\mu$

$$\frac{g_\mu}{\overline{g}_\mu} \equiv 0 \pmod{p}. \tag{2.7}$$

On the other hand, if $\zeta^{(i)}$ is a character in B_τ which represents all elements of $\mathfrak{H}$ by the unit matrix, we have

$$\frac{g_\mu \zeta^{(i)}(G_\mu)}{z_i} = \frac{g_\mu \zeta_i}{z_i}. \tag{2.8}$$

Since $\overline{g}_\mu\, \zeta^{(i)}(\overline{G}_\mu)/z_i$ is an algebraic integer, it follows from (2.7) and (2.8) that

$$\omega_\tau^*(K_\mu) = \omega_\mu^{(i)} \pmod{\mathfrak{p}} = 0.$$

We again consider a subgroup $\mathfrak{N}$ satisfying (2.2). Denote by $\tilde{B}_1, \tilde{B}_2, \cdots\cdots$ the blocks of characters of $\mathfrak{N}$ for p and by $\tilde{\omega}_\sigma^*$ the linear character of A^* corresponding to $\tilde{B}_\sigma$. $\tilde{\omega}_\sigma^*$ induces a linear character of R^* and therefore induces a linear character ω_τ^* of A^* which vanishes for all elements of T^*. Thus we have

$$\omega_\tau^*(K_\nu) = \sum \tilde{\omega}_\sigma^*(\tilde{K}_\rho) \tag{2.9}$$

where the sum extends over the classes $\tilde{K}_\rho$ of $\mathfrak{N}$ which are contained in K_ν. If the defect group of K_ν is $\mathfrak{H}$, K_ν contains one and only one class of $\mathfrak{N}$ and hence there appears only one $\tilde{K}_\rho$ in the sum in (2.9). The block B_τ associated with ω_τ^* is called *the block of* $\mathfrak{G}$ *determined by the block* $\tilde{B}_\sigma$ *of* $\mathfrak{N}$ ([3]). (2.9) yields immediately

$$h \leqq \tilde{d}_\sigma \leqq d_\tau \tag{2.10}$$

where d_τ and $\tilde{d}_\sigma$ are the defects of B_τ and $\tilde{B}_\sigma$, respectively.

4) "i in B_τ" means "$\zeta^{(i)}$ in B_τ".

〔2.3〕 *Let* $\{\mathfrak{H}^{(\alpha)}\}$ *be a complete system of representatives for the classes of conjugate subgroups of order* p^d *in* $\mathfrak{G}$ *and* $\mathfrak{N}^{(\alpha)}$ *the normalizer* $\mathfrak{N}(\mathfrak{H}^{(\alpha)})$ *of* $\mathfrak{H}^{(\alpha)}$. *Then*

1) *All blocks of* $\mathfrak{N}^{(\alpha)}$ *have at least the defect* d. *Each block of defect* d *of* $\mathfrak{N}^{(\alpha)}$ *determines a block of defect* d *of* $\mathfrak{G}$ *and the different blocks of defect* d *of* $\mathfrak{N}^{(\alpha)}$ *determine different blocks of* $\mathfrak{G}$.

2) *Each block of defect* d *of* $\mathfrak{G}$ *is determined by a block of defect* d *of one and only one* $\mathfrak{N}^{(\alpha)}$. (The p-subgroup $\mathfrak{H}^{(\alpha)}$ corresponding to this $\mathfrak{N}^{(\alpha)}$ is called *the defect group* of the block of $\mathfrak{G}$, which is uniquely determined by the block up to conjugacy (〔3〕).)

PROOF. 1) Let $\mathfrak{H}=\mathfrak{H}^{(\alpha)}$ be any subgroup in $\{\mathfrak{H}^{(\alpha)}\}$ and $\tilde{B}_\sigma$ a block of defect d of $\mathfrak{N}=\mathfrak{N}^{(\alpha)}$. There exists a p-regular class $\tilde{K}_\rho$ of defect d such that $\tilde{\omega}_\sigma^*(\tilde{K}_\rho)\not\equiv 0$. Since $\mathfrak{H}$ is the defect group of $\tilde{K}_\rho$, the class K_ν containing $\tilde{K}_\rho$ also has the defect group $\mathfrak{H}$. If B_τ is the block determined by $\tilde{B}_\sigma$, then $\omega_\tau^*(K_\nu)=\tilde{\omega}_\sigma^*(\tilde{K}_\rho)\not\equiv 0$. It follows from 〔1.3〕 and (2.10) that the defect of B_τ is d.

The last part of 1) is readily seen from 〔1.4〕.

2) Let B_τ be a block of defect d. There exists a p-regular class K_ν of defect d such that $\omega_\tau^*(K_\nu)\not\equiv 0$. Let $\mathfrak{H}=\mathfrak{H}^{(\alpha)}$ be the defect group of K_ν. If K_μ is any class belonging to T^*, then by (2.4) $a_{\mu\nu\lambda}\equiv 0 \pmod{p}$ for the classes K_λ such that $\mathfrak{H}_\lambda \subsetneqq \mathfrak{H}_\mu \cap \mathfrak{H}$. Since $\mathfrak{H}_\mu \cap \mathfrak{H}$ is a proper subgroup of $\mathfrak{H}$, 〔1.3〕 yields $\omega_\tau^*(K_\mu)\,\omega_\tau^*(K_\nu)=0$ and hence

$$\omega_\tau^*(K_\mu)=0. \tag{2.11}$$

Thus we can consider ω_τ^* as a linear character of R^* for $\mathfrak{N}=\mathfrak{N}^{(\alpha)}$. Then there exists a linear character $\tilde{\omega}_\sigma^*$ of $\tilde{\Lambda}^*$ which coincides with ω_τ^* on R^*.[5] The block $\tilde{B}_\sigma$ of $\mathfrak{N}$ associated with $\tilde{\omega}_\sigma^*$ determines B_τ and the defect of $\tilde{B}_\sigma$ is d because of 〔2.1〕 and (2.10).

The remaining part of the proof follows immediately from 〔1.4〕 and (2.11).

From the proof of 〔2.3〕 we obtain the following 〔2.4〕 and 〔2.5〕.

〔2.4〕 *A subgroup of order* p^d *of* $\mathfrak{G}$ *is the defect group of a block* B_τ *of defect* d, *if and only if it is the defect group of a* p*-regular class* K_ν *of defect* d *such that* $\omega_\tau^*(K_\nu)\not\equiv 0$.

〔2.5〕 *If the defect group of* K_ν *does not contain the defect group of* B_τ, *then*

$$\omega_\tau(K_\nu)=0.$$

〔2.6〕 *If* $\mathfrak{H}$ *is a normal* p*-subgroup of* $\mathfrak{G}$, *then the defect group of each block* B_τ *contains* $\mathfrak{H}$.

PROOF. Let $\mathfrak{D}_\tau$ be the defect group of B_τ. $\mathfrak{D}_\tau$ is the defect group of a p-regular class K_ν such that $\omega_\tau^*(K_\nu)\not\equiv 0$. Hence 〔2.2〕 yields $\mathfrak{H}\subseteq\mathfrak{D}_\tau$.

[5] See 〔6〕, §1.

〔2.7〕 *Let $\mathfrak{H}$ be a subgroup of order p^h of $\mathfrak{G}$ and $\mathfrak{N}$ a subgroup of $\mathfrak{G}$ satisfying* (2.2). *A block B_τ with the defect group $\mathfrak{D}_\tau$ is determined by a block of $\mathfrak{N}$, if and only if $\mathfrak{H} \subseteq \mathfrak{D}_\tau$. If B_τ is determined by a block $\tilde{B}_\sigma$ of $\mathfrak{N}$ with the defect group $\tilde{\mathfrak{D}}_\sigma$, then $\mathfrak{H} \subseteq \tilde{\mathfrak{D}}_\sigma \subseteq \mathfrak{D}_\tau$.*

PROOF. If $\mathfrak{D}_\tau \supseteq \mathfrak{H}$, then by virtue of 〔2.5〕 B_τ can be determined by a block of $\mathfrak{N}$. If B_τ is determined by $\tilde{B}_\sigma$, then a p-regular class K_ν with the defect group $\mathfrak{D}_\tau$ such that $\omega_\tau^*(K_\nu) \neq 0$ must contain at least one p-regular class $\tilde{K}_\rho$ of $\mathfrak{N}$ such that $\tilde{\omega}_\sigma^*(\tilde{K}_\rho) \neq 0$ because of (2.9). Hence from 〔2.5〕 we have $\mathfrak{D}_\tau \supseteq \tilde{\mathfrak{D}}_\sigma$.

Combining 〔2.5〕 with 〔1.4〕, we obtain

〔2.8〕 *Two characters $\zeta^{(i)}$ and $\zeta^{(j)}$ belonging to blocks with the defect group $\mathfrak{D}$ appear in the same block, if and only if*

$$\omega_\nu^{(i)} \equiv \omega_\nu^{(j)} \pmod{\mathfrak{p}}$$

for all p-regular classes K_ν with the defect group $\mathfrak{D}$.

From 〔2.1〕 and 〔2.3〕 or from 〔2.4〕 we have

〔2.9〕 *If $\mathfrak{H}$ is not a maximal normal p-subgroup of $\mathfrak{N}(\mathfrak{H})$, then $\mathfrak{H}$ cannot be the defect group of any block B_τ.*

〔2.10〕 *If $\mathfrak{G}$ contains a normal p-subgroup $\mathfrak{H}$ and if $\mathfrak{C}(\mathfrak{H})$ is also a p-group, then $\mathfrak{G}$ possesses only one block and the defect of this block is a.*

PROOF. Let $\mathfrak{D}_\tau$ be the defect group of a block B_τ. Since $\mathfrak{D}_\tau$ contains $\mathfrak{H}$, $\mathfrak{C}(\mathfrak{D}_\tau)$ is contained in the p-group $\mathfrak{C}(\mathfrak{H})$. Hence $\mathfrak{C}(\mathfrak{D}_\tau)$ does not contain any p-regular class except K_1 which consists of the unit element. Therefore $\mathfrak{D}_\tau$ is a p-Sylow subgroup of $\mathfrak{G}$. Since $\omega^*(K_1) = 1$ holds for any linear character ω^* of A^*, by 〔1.4〕 $\mathfrak{G}$ possesses only one block of defect a.

3.[6] Let $e_1, e_2, \cdots\cdots, e_n$ be the primitive idempotents of $A = A(\mathfrak{G})$ corresponding to $\zeta^{(1)}, \zeta^{(2)}, \cdots\cdots, \zeta^{(n)}$, respectively. Each e_i is given by

$$e_i = \frac{1}{g} \sum_{\nu=1}^{n} z_i \, \zeta_\nu^{(i)} \, K_\nu \tag{3.1}$$

and we have

$$K_\nu e_i = \omega_\nu^{(i)} e_i \text{ and } \omega^{(i)}(e_i) = 1. \tag{3.2}$$

For each block B_τ we consider the idempotent

$$E = \sum_{i \text{ in } B_\tau} e_i .$$

From (3.1) we have

[6] Cf. 〔7〕.

(3.3) $$E_\tau = \sum_\nu b_\nu^{(\tau)} K_\nu$$

where $b_\nu^{(\tau)}$ are $\mathfrak{p}$-integers. For all p-singular classes K_σ

$$b_\sigma^{(\tau)} = 0.$$

Put $E_\tau \pmod{\mathfrak{p}} = E_\tau^*$ and $b_\nu^{(\tau)} \pmod{\mathfrak{p}} = b_\nu^{(\tau)*}$. We then have

(3.4) $$E_\tau^* = \sum_\nu b_\nu^{(\tau)*} K_\nu ,$$

(3.5) $$\omega_\tau^*(E_\tau^*) = \sum_\nu b_\nu^{(\tau)*}(K_\nu) = 1$$

and

(3.6) $$1 = \sum_\tau E_\tau^* .$$

Hence we see that the elements E_τ^* are the mutually orthogonal primitive idempotents of $A^* = A^*(\mathfrak{G})$.

Let denote by $\mathfrak{D}_\tau$ the defect group of B_τ.

〔3.1〕 *Each E_τ^* is a linear combination of p-regular classes K_ν with $\mathfrak{H}_\nu \subseteq \mathfrak{D}_\tau$.*

PROOF. If K_ν is a class such that $b_\nu^{(\tau)*} \neq 0$, then by (3.4) B_τ is determined by a block of $\mathfrak{N}(\mathfrak{H}_\nu)$ since for $\mathfrak{H} = \mathfrak{H}_\nu$ the elements $E_\sigma^* \pmod{T^*} \neq 0$ form the complete system of mutually orthogonal primitive idempotents of A^*/T^*. Hence 〔2.7〕 yields $\mathfrak{H}_\nu \subseteq \mathfrak{D}_\tau$.

〔3.2〕 *There exists at least one p-regular class K_ν with the defect group $\mathfrak{D}_\tau$ such that both $b_\nu^{(\tau)*}$ and $\omega_\tau^*(K_\nu)$ do not vanish.*

PROOF. By 〔1.3〕 we have

$$\omega_\tau^*(K_\mu) = 0$$

for all classes K_μ with $\mathfrak{H}_\mu \subset \mathfrak{D}_\tau$. Therefore, by 〔3.1〕 and (3.5) there exists at least one p-regular class K_ν with $\mathfrak{H}_\nu = \mathfrak{D}_\tau$ such that

$$\omega_\tau^*(K_\nu) \neq 0 \quad \text{and} \quad b_\nu^{(\tau)*} \neq 0.$$

We now consider the elements $K_\alpha E_\tau$ and put

(3.7) $$K_\alpha E_\tau = \sum a_{\alpha\gamma}^{(\tau)} K_\gamma .$$

For any p-regular class K_α the above sum extends over the p-regular classes only. If the p-regular classes $K_{\tau,1}$, $K_{\tau,2}$, ……, K_{τ,y_τ} are associated with B_τ in the sense of 〔1.1〕, then the elements $K_{\tau,1}E_\tau$, $K_{\tau,2}E_\tau$, ……, $K_{\tau,y_\tau}E_\tau$ are linearly independent (mod $\mathfrak{p}$). We show this. From (3.1) and (3.2) we have

$$(K_{\tau,1}E_\tau \quad K_{\tau,2}E_\tau \cdots\cdots \quad K_{\tau,y_\tau}E_\tau) = (K_1 \quad K_2 \cdots\cdots \quad K_k)A_\tau^{(0)}$$

where

$$A_\tau^{(\mathfrak{P})} = (\varphi_{\nu,\mu}^{(\kappa)})' C_\tau (\varphi_{\tau,l}^{(\kappa)}) \begin{pmatrix} n_{\tau,1}^{-1} & & & 0 \\ & n_{\tau,2}^{-1} & & \\ & & \ddots & \\ 0 & & & n_{\tau,y_\tau}^{-1} \end{pmatrix}^{7)}$$

$$(\varphi^{(\kappa)} \text{ in } B_\tau;\ \nu = 1, 2, \cdots\cdots, k;\ l = 1, 2, \cdots\cdots, y_\tau).$$

Hence by [1.1] the rank of $A_\tau^{(\mathfrak{P})} (\mathrm{mod}\ \mathfrak{p})$ is y_τ.

Conversely if the p-regular classes $K'_{\tau,1}, K'_{\tau,2}, \cdots\cdots, K'_{\tau,y'_\tau}$ are associated with B_τ in such a way that each p-regular class is associated with one and only one block and y'_τ elements $K'_{\tau,l} E_\tau$ are linearly independent $(\mathrm{mod}\ \mathfrak{p})$, then we have $y'_\tau = y_\tau$ and y'_τ p-regular classes $K'_{\tau,l}$ can be associated with B_τ in the sense of [1.1].

Let P be an element of order p^h belonging to the center of $\mathfrak{G}$. Any class K_α which contains elements with the p-factor[8] P is of the form $K_\alpha = PK_\beta$ where K_β is a p-regular class. Therefore, for any class K_α which consists of elements with the p-factor P the sum in (3.7) extends over the classes K_γ which consist of elements with the p-factor P only.[9] Further, if the p-regular classes $K_{\tau,1}, K_{\tau,2}, \cdots\cdots, K_{\tau,y_\tau}$ are associated with B_τ in the sense of [1.1], then y_τ elements $PK_{\tau,l} E_\tau$ are linearly independent $(\mathrm{mod}\ \mathfrak{p})$.

Let $\mathfrak{H}$ be any p-subgroup of $\mathfrak{G}$ and consider (2.5) and (2.6) for $\mathfrak{N} = \mathfrak{N}(\mathfrak{H})$. Further let $\tilde{P}_0 = 1, \tilde{P}_1, \tilde{P}_2, \cdots\cdots$ be a complete system of representatives for the classes of conjugate elements of $\mathfrak{N}$ whose orders are powers of p and $\tilde{E}_\sigma^{\mathfrak{P}}$ the primitive idempotent of $A^{\mathfrak{P}}(\mathfrak{N})$ corresponding to a block $\tilde{B}_\sigma$ of $\mathfrak{N}$. We then show the following: *If* $\tilde{K}_1^{(\mathfrak{P})}, \tilde{K}_2^{(\mathfrak{P})}, \cdots\cdots$ *are the classes of* $\mathfrak{N}$ *which have the defect group* $\mathfrak{H}$ *and contain elements with the* p*-factor* $\tilde{P}_j$, *then we have*

$$\tilde{K}_\lambda^{(\mathfrak{P})} \tilde{E}_\sigma^{\mathfrak{P}} = \sum a_{\lambda\mu}^{(\sigma)\mathfrak{P}} \tilde{K}_\mu^{(\mathfrak{P})}. \tag{3.8}$$

Since $\mathfrak{H}$ is the defect group of $\tilde{K}_\lambda^{(\mathfrak{P})}$, the p-factor of any element in $\tilde{K}_\lambda^{(\mathfrak{P})}$ belongs to $\mathfrak{H} \cap \mathfrak{C}(\mathfrak{H})$ contained in the center of $\mathfrak{C}(\mathfrak{H})$. Evidently $\tilde{K}_\lambda^{(\mathfrak{P})}$ is a sum of complete classes of $\mathfrak{C}(\mathfrak{H})$ which consist of elements with the p-factors conjugate to $\tilde{P}_j$ in $\mathfrak{N}$. Further $\tilde{E}_\sigma^{\mathfrak{P}}$ belongs to $A^{\mathfrak{P}}(\mathfrak{C}(\mathfrak{H}))$[10] and is a sum of primitive idempotents of $A^{\mathfrak{P}}(\mathfrak{C}(\mathfrak{H}))$. Therefore applying the above result, the element $\tilde{K}_\lambda^{(\mathfrak{P})} \tilde{E}_\sigma^{\mathfrak{P}}$ of $A^{\mathfrak{P}}(\mathfrak{C}(\mathfrak{H}))$ is a linear combination of the classes of $\mathfrak{C}(\mathfrak{H})$ which consist of elements with the p-factors conjugate to $\tilde{P}_j$ in $\mathfrak{N}$.

7) $\varphi_{\tau,l}^{(\kappa)}$ denotes the value of $\varphi^{(\kappa)}$ for the class $K_{\tau,l}$ and $n_{\tau,l}$ denotes the order of the normalizer $\mathfrak{N}(G_{\tau,l})$ of an element $G_{\tau,l}$ in $K_{\tau,l}$.

8) Let Q be an element of $\mathfrak{G}$ such that the order of Q is a power of p. If G is an element of the form QR where R is a p-regular element of $\mathfrak{N}(Q)$, then Q is called the *p-factor* of G.

9) From Theorem 2 in [4], we see that this fact remains valid even though P does not belong to the center of $\mathfrak{G}$. But the writer does not know the proof of this theorem.

10) This can be seen immediately from (2.6) and [2.2]. Cf. [7], §3.

Consequently the element $\tilde{K}_\lambda^{(\mathfrak{H})}\tilde{E}_\sigma^{*}$ of $A^*(\mathfrak{N})$ is a linear combination of the classes $\tilde{K}_\rho$ of $\mathfrak{N}$ which contain elements with the p-factor $\tilde{P}_j$. Since every element in each $\tilde{K}_\rho$ belongs to $\mathfrak{C}(\mathfrak{H})$, the defect group $\tilde{\mathfrak{H}}_\rho$ of $\tilde{K}_\rho$ contains $\mathfrak{H}$. On the other hand, by (2.4) $\tilde{\mathfrak{H}}_\rho$ is contained in $\mathfrak{H}$. From the above (3.8) follows.

We now consider the classes $\tilde{K}_\lambda$ of $\mathfrak{N}$ with the defect group $\mathfrak{H}$. We then show the following: *It is possible to associate these classes $\tilde{K}_\lambda$ with the blocks $\tilde{B}_\sigma$ in such a way that each class $\tilde{K}_\lambda$ is associated with one and only one block $\tilde{B}_\sigma$ and for the classes $\tilde{K}_{\sigma,l}$ ($l = 1, 2, \cdots\cdots, \tilde{r}_\sigma$) associated with $\tilde{B}_\sigma$ $\tilde{K}_{\sigma,l}\tilde{E}_\sigma^{*}$ are linearly independent.*

By (3.8) it is sufficient to consider only the classes $\tilde{K}_\lambda^{(\mathfrak{H})}$ for a fixed $\tilde{P}_j$. Since $\tilde{P}_j$ belongs to the center of $\mathfrak{C}(\mathfrak{H})$, $\tilde{\mathfrak{N}}(\tilde{P}_j) = \mathfrak{N}(\tilde{P}_j) \cap \mathfrak{N}$ contains $\mathfrak{C}(\mathfrak{H})$ and hence every element in any $\tilde{K}_\lambda^{(\mathfrak{H})}$ belongs to $\tilde{\mathfrak{N}}(\tilde{P}_j)$. Evidently the elements in each $\tilde{K}_\lambda^{(\mathfrak{H})}$ with the p-factor $\tilde{P}_j$ form a complete class $\hat{K}_\lambda^{(\mathfrak{H})}$ of $\tilde{\mathfrak{N}}(\tilde{P}_j)$ and the defect group of $\hat{K}_\lambda^{(\mathfrak{H})}$ is also $\mathfrak{H}$. Conversely if a class of $\tilde{\mathfrak{N}}(\tilde{P}_j)$ has the defect group $\mathfrak{H}$ and consists of elements with the p-factor $\tilde{P}_j$, then this class is contained in one class $\tilde{K}_\lambda^{(\mathfrak{H})}$. On the other hand, since $\tilde{E}_\sigma^{*}$ belongs to $A^*(\mathfrak{C}(\mathfrak{H}))$, it belongs to $\Gamma^*(\tilde{\mathfrak{N}}(\tilde{P}_j))$ and hence to $A^*(\tilde{\mathfrak{N}}(\tilde{P}_j))$. Denoting by $\hat{B}^{(\sigma)}$ the collection of blocks of $\tilde{\mathfrak{N}}(\tilde{P}_j)$ corresponding to the idempotent $\tilde{E}_\sigma^{*}$ of $A^*(\tilde{\mathfrak{N}}(\tilde{P}_j))$, we can associate the classes $\hat{K}_\lambda^{(\mathfrak{H})}$ with the collections $\hat{B}^{(\sigma)}$ in such a way that each $\hat{K}_\lambda^{(\mathfrak{H})}$ is associated with one and only one collection $\hat{B}^{(\sigma)}$ and for the classes $\hat{K}_{\sigma,l}$ associated with $\hat{B}^{(\sigma)}$ $\hat{K}_{\sigma,l}\tilde{E}_\sigma^{*}$ are linearly independent. As is easily seen, the corresponding elements $\tilde{K}_{\sigma,l}\tilde{E}_\sigma^{*}$ are linearly independent.

If $\tilde{s}_\sigma^{(\mathfrak{H})}$ is the number of the classes $\tilde{K}_{\sigma,l}$ which contain elements with the p-factor $\tilde{P}_j$, then $\tilde{s}_\sigma^{(\mathfrak{H})}$ depends only on $\mathfrak{H}$, $\tilde{B}_\sigma$ and $\tilde{P}_j$.

These facts remain valid, if $\tilde{B}_\sigma$ are collections of blocks of $\mathfrak{N}$ such that each block of $\mathfrak{N}$ belongs to one and only one collection. In this case the classes associated with a collection can be associated with the blocks belonging to this collection in the above sense.

Let $\{\mathfrak{H}\}$ be a complete system of representatives for the classes of conjugate subgroups of orders p^h ($h = 0, 1, 2, \cdots\cdots$) in $\mathfrak{G}$. If the defect group of a block B_τ contains a subgroup $\mathfrak{H}$ in $\{\mathfrak{H}\}$, then for $\mathfrak{N} = \mathfrak{N}(\mathfrak{H})$ the element $\tilde{E}^{(\tau)*}$ of R^* corresponding to E_τ^{*} (mod T^*) in (2.5) is a sum of primitive idempotents $\tilde{E}_\sigma^{*}$ of $A^*(\mathfrak{N}(\mathfrak{H}))$. The blocks $\tilde{B}_\sigma$ belonging to these idempotents $\tilde{E}_\sigma^{*}$ form the collection $\tilde{B}^{(\tau)}(\mathfrak{H})$ of blocks of $\mathfrak{N}(\mathfrak{H})$ which determine the block B_τ.

For each subgroup $\mathfrak{H}$ in $\{\mathfrak{H}\}$ consider the classes $\tilde{K}_{\tau,l}(\mathfrak{H})$ of $\mathfrak{N}(\mathfrak{H})$ which have the defect group $\mathfrak{H}$ and are associated with $B^{(\tau)}(\mathfrak{H})$, $l = 1, 2, \cdots\cdots, r_\tau(\mathfrak{H})$. Then we have

[3.3] *If $K_{\tau,l}(\mathfrak{H})$ is the class of $\mathfrak{G}$ containing $\tilde{K}_{\tau,l}(\mathfrak{H})$, all the classes of $\mathfrak{G}$ are given by*

(3.9) $K_{\tau,l}(\mathfrak{H})$ $(l=1,2,\cdots\cdots,r_\tau(\mathfrak{H});\ \tau=1,2,\cdots\cdots,s;\ \mathfrak{H}\ in\ \{\mathfrak{H}\})$

and the elements

(3.10) $K_{\tau,l}(\mathfrak{H})E_\tau$ $(l=1,2,\cdots\cdots,r_\tau(\mathfrak{H});\ \tau=1,2,\cdots\cdots,s;\ \mathfrak{H}\ in\ \{\mathfrak{H}\})$

are linearly independent (*mod* $\mathfrak{p}$). *Conversely let* $K'_{\tau,l}$ *be the classes of* $\mathfrak{G}$ *associated with* B_τ *in such a way that each class* K_ν *is associated with one and only one block of* $\mathfrak{G}$ *and* $K'_{\tau,l}E_\tau$ *are linearly independent* (*mod* $\mathfrak{p}$). *Then this selection of classes of* $\mathfrak{G}$ *can be obtained as above.*[11]

PROOF. Arrange all subgroups $\mathfrak{H}$ in $\{\mathfrak{H}\}$ as follows: $\mathfrak{H}^{(1)}$, $\mathfrak{H}^{(2)}$, $\cdots\cdots$, where the order of $\mathfrak{H}^{(i)}$ is not less than that of $\mathfrak{H}^{(j)}$ if $i<j$. Considering (2.5) for $\mathfrak{H}=\mathfrak{H}^{(i)}$ and $\mathfrak{N}=\mathfrak{N}(\mathfrak{H})$, $i=1,2,\cdots\cdots$, we see that the elements in (3.10) are linearly independent (mod $\mathfrak{p}$).

We now prove the converse. If for $\mathfrak{N}=\mathfrak{N}(\mathfrak{H})$ we consider (2.5), then $K'^{0}_{\tau,l}(\neq 0)$ form a K^*-basis of R^* and hence for the classes $K'^{0}_{\tau,l}\neq 0$ $K'^{0}_{\tau,l}\bar{E}^{(\tau)*}$ also form a K^*-basis of R^*. In particular for the classes $K'_{\tau,l}$ with the defect group $\mathfrak{H}$ $K'_{\tau,l}E^{(\tau)*}$ are linearly independent. This completes the proof.

Let $P_0=1$, P_1, P_2, $\cdots\cdots$ be a complete system of representatives for the classes of conjugate elements of order p^h $(h=0,1,2,\cdots\cdots)$ in $\mathfrak{G}$.

[3.4] *If, in the notation of* [3.3], *we denote by* $s_\tau^{(j)}(\mathfrak{H})$ *the number of classes* $K_{\tau,l}(\mathfrak{H})$ *which contain elements with the p-factor* P_j, *then* $s_\tau^{(j)}(\mathfrak{H})$ *depends only on* B_τ, $\mathfrak{H}$ *and* P_j. *In particular both* $s_\tau(\mathfrak{H})(=s_\tau^{(0)}(\mathfrak{H}))$ *and* $r_\tau(\mathfrak{H})(=\sum_j s_\tau^{(j)}(\mathfrak{H}))$ *depend only on* B_τ *and* $\mathfrak{H}$. *Further we have* $\sum_{\mathfrak{H}} r_\tau(\mathfrak{H})=x_\tau$ *and* $\sum_{\mathfrak{H}} s_\tau(\mathfrak{H})=y_\tau$.

[3.5] *The defect groups of the classes associated with* B_τ *in the sense of* [3.3] *are contained in the defect group* $\mathfrak{D}_\tau$ *of* B_τ. *For each* B_τ *exactly one of the p-regular classes associated with* B_τ *has the defect group* $\mathfrak{D}_\tau$.

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[11] Such a selection is identical with that in [3], §5.

ON THE NUMERICAL SOLUTIONS OF SOME INTEGRO-DIFFERENTIAL EQUATIONS UNDER SOME CONDITIONS

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(Received February 28, 1956)

I treated of a method to find the numerical solutions of some integro-differential equations which are in the form

$$f(x,y(x),y'(x),y''(x)) + \int_a^b \varphi(x,\xi,y(\xi),y'(\xi),y''(\xi))d\xi = 0$$

under some conditions that the values of y and y' at several values of x must be satisfied given functional relations, and few examples were given.

1. Introduction.

There are many methods for the numerical solution of a differential equation with *initial conditions*, but when the conditions are given other forms (e.g. the solution curve must pass a given point and intersect to a given curve with a given angle), it seems that there is no systematic method and we often employ so-called *trial and errors method*. But this method is very laborious and we feel the necessity of other convenient one. I researched on this problem and reported some results in the Kumamoto Journal of Science[1]. In this paper, I considered similar problem referring to some integro-differential equations, and by substituting a set of *difference equations* for this equation we can find the numerical solution of them.

2. General Theory.

Let $f(x,y,y',y'')$, $\varphi(x,\xi,y,y',y'')$ be given functions and a, b be given constants and consider an integro-differential equation

$$f(x,y(x),y'(x),y''(x)) + \int_a^b \varphi(x,\xi,y(\xi),y'(\xi),y''(\xi))\,d\xi = 0 \quad \cdots\cdots \quad (1)$$
$$a \leqq x \leqq b$$

and let us suppose the existence of its solution.

At first, put $(b-a)/m=h$, $x_i=\xi_i=a+ih$, $y_i=y(x_i)$, $y_i'=y'(x_i)$, $y_i''=y''(x_i)$ and by using the approximate formulas for definite integrals[2], (1) becomes

$$f(x,\ y(x),\ y'(x),\ y''(x)) + \sum_{i=0}^{m} R_i \varphi(x,\ \xi_i,\ y_i, y_i',\ y_i'') = 0 \quad \cdots\cdots\cdots \quad (2)$$

where R'S are some known constants which are determined from the approximate formulas

which are emploied. Put $x = x_s$ in the equation (2), then we get

$$f(x_s, y_s, y'_s, y''_s) + \sum_{l=0}^{m} R_l \varphi(x_s, \xi_l, y_l, y'_l, y''_l) = 0 \quad \cdots\cdots (3)$$
$$s = 0, 1, 2, \cdots\cdots, m.$$

Next, by the approximate formulas for differential coefficients[3], we get

$$\left.\begin{aligned} y'_s &= \sum_{i=0}^{p} B_{i,j} y_{i+s-j}, \quad y''_s = \sum_{i=0}^{p} C_{i,k} y_{i+s-k} \\ y'_l &= \sum_{i=0}^{p} B_{i,u} y_{i+l-u}, \quad y''_l = \sum_{i=0}^{p} C_{i,v} y_{i+l-v} \end{aligned}\right\} \quad \cdots\cdots (4)$$

where j, k, u, v are any integer provided that they satisfy that $0 \leqq j, k, u, v \leqq s$, $s-(m-p) \leqq j, k, u, v$ and we may determine them conveniently. Substituting (4) in (3), we obtain $m+1$ simultaneous equations with respect to $y_0, y_1, y_2, \cdots\cdots, y_m$ and if we can solve these equations numerically, our problem becomes to be solved.

At beginning, if $f(x, y, y', y'')$ involves y'' actually, (3) can be transformed to the form

$$y''_s = \Phi\left(x_s, y_s, y'_s, \sum_{l=0}^{m} R_l \varphi(x_s, \xi_l, y_l, y'_l, y''_l)\right) \quad \cdots\cdots (5)$$
$$s = 0, 1, 2, \cdots\cdots, m$$

and by the properties of $C_{i,j}$ these simultaneous equations are indeterminate and in order to solve them, two conditions must be added[4]. As these conditions we take

$$\left.\begin{aligned} g_1(y_0, y_1, y_2, \cdots\cdots, y_m, y'_0, y'_1, y'_2, \cdots\cdots, y'_m) &= 0 \\ g_2(y_0, y_1, y_2, \cdots\cdots, y_m, y'_0, y'_1, y'_2, \cdots\cdots, y'_m) &= 0 \end{aligned}\right\} \quad \cdots\cdots (6)$$

where g_1, g_2 are given functions. Therefore we must take (6) in places of any two equations of $(m+1)$ equations which are made by putting $s = 0, 1, 2, \cdots\cdots, m$ in (3), and substitute the approximate formulas for $y'_0, y'_1, \cdots\cdots, y'_m$ in (6) and then solve these equations simultaneously with respect to $y_0, y_1, y_2, \cdots\cdots, y_m$.

Similarly if $f(x, y, y', y'')$ has not y'' and involves y' actually, we must take a condition in place of any one equation of $(m+1)$ equations which are made by putting $s = 0, 1, 2, \cdots\cdots, m$ in (3), and as this condition we take

$$g(y_0, y_1, y_2, \cdots\cdots, y_m) = 0 \quad \cdots\cdots (7).$$

Lastly, if $f(x, y, y', y'')$ does not involve y', y'' both, no condition is necessary.

Accordingly, to solve the numerical solutions of the integro-differential equation (1) can be transformed to solve some simultaneous equations numerically.

If $f(x, y, y', y'')$, $\varphi(x, \xi, y, y', y'')$ and the conditions (6), (7) are together linear with respect to y, y', y'', the corresponding simultaneous equations also linear with respect to the unknowns, therefore these simultaneous equations can be solved exactly. Next, if at

least, one of $f(x, y, y', y'')$, $\varphi(x, \xi, y, y', y'')$ or equation (6) or (7) is not linear, the corresponding simultaneous equations are not linear also, and in most cases, it is impossible to solve them exactly, but their numerical solutions can be calculated at any accuracy by suitable method (e. g. iterative method or Newton-Raphson's method etc.).

3. When $\varphi(x, \xi, y, y', y'')$ is Linear Form

When $\varphi(x, \xi, y, y', y'')$ is linear form with respect to y, y', y'', putting

$$\varphi(x, \xi, y, y', y'') = K(x, \xi)\, y(\xi) + L(x, \xi) y'(\xi) + M(x, \xi) y''(\xi) \cdots\cdots (8)$$

gives

$$\int_a^b L(x, \xi) y'(\xi) d\xi = L(x, b) y_m - L(x, a) y_0 - \int_a^b \frac{\partial}{\partial \xi} L(x, \xi) y(\xi) d\xi$$

$$\int_a^b M(x, \xi) y''(\xi) d\xi = M(x, b) y'_m - M(x, a) y'_0 - \int_a^b \frac{\partial}{\partial \xi} M(x, \xi) y'(\xi) d\xi$$

$$= M(x, b) y'_m - M(x, a) y'_0 - \frac{\partial}{\partial \xi} M(x, b) y_m + \frac{\partial}{\partial \xi} M(x, a) y_0$$

$$+ \int_a^b \frac{\partial^2}{\partial \xi^2} M(x, \xi) y(\xi) d\xi$$

and the equation (1) becomes

$$f\Big(x, y(x), y'(x), y''(x)\Big) + \Big\{L(x, b) - \frac{\partial}{\partial \xi} M(x, b)\Big\} y_m - \Big\{L(x, a)$$

$$- \frac{\partial}{\partial \xi} M(x, a)\Big\} y_0 + M(x, b) y'_m - M(x, a) y'_0 + \int_a^b \Big\{K(x, \xi) - \frac{\partial}{\partial \xi} L(x, \xi)$$

$$+ \frac{\partial^2}{\partial \xi^2} M(x, \xi)\Big\} y(\xi) d\xi = 0 \cdots\cdots (9).$$

In the practical procedure, the equation (9) is less laborious than the equation (1), especially when the values of $y(a)$, $y'(b)$ etc. are given.

4. Errors

Concerning the errors of the approximate formulas, we have following inequalities[5]

$$\left.\begin{aligned} |R_{p,j}| &\leqq \frac{|h^{p+2}|}{(p+1)!}\left[\sum_{i=1}^{p}\left|\frac{A_{i,j}}{h}\right| \cdot i^{p+1} + \frac{i^{p+2}}{p+2}\right] G \qquad \text{for } F_j \\ |R_{p,j}| &\leqq \frac{|h^{p}|}{(p+1)!}\left[\sum_{i=1}^{p}|hB_{i,j}| \cdot i^{p+1} + (p+1) j^{p}\right] G \qquad \text{for } y'_j \\ |R_{p,j}| &\leqq \frac{|h^{p-1}|}{(p+1)!}\left[\sum_{i=1}^{p}|h^2 C_{i,j}| \cdot i^{p+1} + p(p+1) j^{p-1}\right] G \quad \text{for } y''_j \end{aligned}\right\} \cdots\cdots (10)$$

where G is the upper bound of $|y^{(p+1)}(x)|$ when $a \leqq x \leqq b$.

Next, the errors of the solutions of the corresponding simultaneous equations can be estimated if we do not mind the trouble of calculating, but really the procedure is very troublesome and moreover the function $y(x)$ is unknown function, hence generally it is difficult to determine the value of G accurately. But if necessary, there are some ways to assume the approximate value of G from the numerical solutions of $y(x)$ obtained by above method. One way is the successive application of the approximate formulas for y_j' or y_j'', that is

$$y_s' = \sum_{i=0}^{p} B_{i,j} y_{i+s-j}, \quad y_s'' = \sum_{i=0}^{p} C_{i,j} y_{i+s-j}$$

$$y_s''' = \sum_{i=0}^{p} C_{i,j} y_{i+s-j}', \quad y_s^{(iv)} = \sum_{i=0}^{p} C_{i,j} y_{i+s-j}''$$

.....

and put the greatest value of $|y_s^{(p+1)}|$ equals to G. But as $B_{i,j}$, $C_{i,j}$ have h^{-1}, h^{-2} as their factors respectively, when we continue this method many times, their errors become very large. The second way is as follows. In the equation (2), by applying the approximate formulas for y_l', y_l'', the term $\sum_{l=0}^{m} R_l \varphi(x, \xi_l, y_l, y_l', y_l'')$ becomes to a known function of x and let it be $\Psi(x)$, then we get

$$f(x, y(x), y'(x), y''(x)) + \Psi(x) = 0$$

$$\therefore \quad f_x(x, y(x), y'(x), y''(x)) + f_y(x, y(x), y'(x), y''(x))y'(x)$$
$$+ f_{y'}(x, y(x), y'(x), y''(x))y''(x) + f_{y''}(x, y(x), y'(x), y''(x))y'''(x) + \Psi'(x) = 0$$

and putting $x = x_s$ gives

$$f_x(x_s, y_s, y_s', y_s'') + f_y(x_s, y_s, y_s', y_s'')y_s' + f_{y'}(x_s, y_s, y_s', y_s'')y_s''$$
$$+ f_{y''}(x_s, y_s, y_s', y_s'')y_s''' + \Psi'(x_s) = 0$$

By calculating the values of y_s', y_s'' from the approximate formulas and substituting them into above equation, we can calculate the value of y_s''', recurring this procedure, we can find the values of $y_s^{(iv)}$, $y_s^{(v)}$, ..., successively. Generally, the latter gives very smaller errors than the former, and especially, when $f(x, y, y', y'')$ is linear with respect to y, y', y'' and their coefficients are constants or simple forms of x, the latter is less laborious.

5. Examples

Example 1. Solve the equation

$$y'(x) + y(x) - \frac{1}{6}(12x^2 - 11) + \int_0^1 \{(x+\xi)y'(\xi) - (x\xi - 2)y(\xi)\} d\xi = 0 \quad \cdots\cdots \text{(i)}$$

$$y(x) = x^2 + 1$$

under the condition $y(1)=0$.

Solution :—

At first

$$\int_0^1\left\{(x+\xi)y(\xi)-(x\xi-2)y(\xi)\right\}d\xi=-xy(0)-\int_0^1(x\xi-1)y(\xi)d\xi$$

therefore (i) becomes

$$y'(x)+y(x)-\frac{1}{\sqrt{6}}(12x^2-11)-xy(0)-\int_0^1(x\xi-1)y(\xi)d\xi=0 \quad \cdots\cdots\cdots \quad \text{(ii)}.$$

Putting $a=0$, $b=1$, $m=10$ $\therefore$ $h=0.1$, $x_i=\xi_i=0.1i$ and applying the *Simpson's rule*

$$\begin{aligned}I(x)=\frac{1}{30}\Big\{&(0-1)y_0+4(0.1x-1)y_1+2(0.2x-1)y_2+4(0.3x-1)y_3\\&+2(0.4x-1)y_4+4(0.5x-1)y_5+2(0.6x-1)y_6+4(0.7x-1)y_7\\&+2(0.8x-1)y_8+4(0.9x-1)y_9+(x-1)y_{10}\Big\} \quad \cdots\cdots\cdots \quad \text{(iii)}\end{aligned}$$

give

$$y'(x)+y(x)-\frac{1}{\sqrt{6}}(12x^2-11)-xy_0-I(x)=0 \quad \cdots\cdots\cdots \quad \text{(iv)}$$

and by putting $x=x_s$ in (iv), we have

$$y'_s+y_s-\frac{1}{\sqrt{6}}(0.12s^2-11)-0.1sy_1-I_s=0 \quad \cdots\cdots\cdots \quad \text{(v)}$$

where $I_s=I(x_s)$, $s=0, 1, 2, \cdots\cdots, 9$.

While by the approximate formulas for differential coefficients[6], we have ($p=3$ and $y_{10}=0$)

$$\left.\begin{aligned}
y'_0&=\frac{1}{h}\left(-\frac{11}{6}y_0+3y_1-\frac{3}{2}y_2+\frac{1}{3}y_3\right)\\
y'_1&=\frac{1}{h}\left(-\frac{1}{3}y_0-\frac{1}{2}y_1+y_2-\frac{1}{6}y_3\right)\\
y'_2&=\frac{1}{h}\left(-\frac{1}{3}y_1-\frac{1}{2}y_2+y_3-\frac{1}{6}y_4\right)\\
y'_3&=\frac{1}{h}\left(-\frac{1}{3}y_2-\frac{1}{2}y_3+y_4-\frac{1}{6}y_5\right)\\
y'_4&=\frac{1}{h}\left(-\frac{1}{3}y_3-\frac{1}{2}y_4+y_5-\frac{1}{6}y_6\right)\\
y'_5&=\frac{1}{h}\left(-\frac{1}{3}y_4-\frac{1}{2}y_5+y_6-\frac{1}{6}y_7\right)\\
y'_6&=\frac{1}{h}\left(-\frac{1}{3}y_5-\frac{1}{2}y_6+y_7-\frac{1}{6}y_8\right)\\
y'_7&=\frac{1}{h}\left(-\frac{1}{3}y_6-\frac{1}{2}y_7+y_8-\frac{1}{6}y_9\right)\\
y'_8&=\frac{1}{h}\left(-\frac{1}{3}y_7-\frac{1}{2}y_8+y_9\right)\\
y'_9&=\frac{1}{h}\left(\frac{1}{6}y_7-y_8+\frac{1}{2}y_9\right)
\end{aligned}\right\} \quad \cdots\cdots\cdots \quad \text{(vi)}$$

By putting $s=0, 1, 2, \cdots, 9$ and substituting (iii), (vi)give us following simultaneous equations.

(vii)

These are the linear equations and we can solve them exactly and the result of my calculation is as follows.

Table. 1.

i	y_i	i	y_i
0	2.44949	6	− 0.19596
1	1.76363	7	− 0.29394
2	1.17576	8	− 0.29394
3	0.68586	9	− 0.19596
4	0.29394	10	0.00000
5	0.00000		

As a matter of fact, $y=\sqrt{6}\,(2x^2-3x+1)$ is the solution of the equation (i) and above values satisfy this expression exactly, namely as $y^{(p)}(x)\equiv 0$, $p\geqq 3$, we get $G=0$ therefore the errors of the approximate formulas are all zero and the simultaneous equations (vii) have no error.

Example. 2.

Solve the equation

$$y''(x)+y(x)+0.005166\,x-0.3328173 + \int_0^{0.5}\Big[(5x-2\xi)y(\xi)+\xi(\xi+x)y'(\xi)+\xi^2(\xi-x)y''(\xi)\Big]d\xi=0 \quad\cdots\cdots\quad \text{(i)}$$

under the conditions

$$y(0.5)=0,\quad y(0)/y(0.3)=2.4567335.$$

Solution :—

At first, as the integrand is linear, (i) becomes

$$y''(x)+y(x)+0.005166\,x-0.3328173 - (0.125-0.25x)y(0.5)-2\int_0^{0.5}(\xi+x)\,y(\xi)\,d\xi=0 \quad\cdots\cdots\quad \text{(ii)}$$

By putting $a=0$, $b=0.5$, $m=5$ we have $h=0.1$, $y_5=0$, $y_0=2.4567335y_3$, and by the approximate formula for definite integral $(p=5)$, we get

$$F_{5.5}=\int_0^{0.5}(\xi+x)\,y(\xi)d\xi=\frac{5h}{288}\Big[19xy_0+75(0.1+x)\,y_1 + 50\,(0.2+x)\,y_2+50\,(0.3+x)\,y_3+75\,(0.4+x)\,y_4\Big].$$

Putting $0.005166=\alpha$, $0.3328173=\beta$, $2.4567335=\gamma$ and substituting them into (ii), we have

$$y''(x)+y(x)+\alpha x-\beta+(0.125-0.25x)y_5+\frac{1}{288}\Big[75(0.1+x)y_1 +50(0.2+x)y_2+\{15+(50+19\gamma)x\}y_3+75(0.4+x)y_4\Big]=0\cdots\cdots \quad \text{(iii)}.$$

Here put $x=x_s$, then we get

$$y_s''+y_s+0.1\alpha s-\beta+(0.125-0.025s)y_s'+\frac{1}{288}\Big[7.5(1+s)y_1$$
$$+5(2+s)y_2+\{15+(5+1.9\gamma)s\}y_3+7.5(4+s)y_4\Big]=0 \quad \cdots\cdots\cdots \text{(iv)}.$$

While by the approximate formulas for y_j', $y_j''(p=5)$, we get

$$\left.\begin{aligned}
y_0' &= 10\Big[\ \frac{5}{4}y_1-\frac{10}{3}y_2+\Big(5-\frac{\gamma}{5}\Big)y_3-5y_4\Big]\\
y_1'' &= 100\Big[-\frac{5}{4}y_1-\frac{1}{3}y_2+\frac{7+5\gamma}{6}y_3-\frac{1}{2}y_4\Big]\\
y_2'' &= 100\Big[\ \frac{4}{3}y_1-\frac{5}{2}y_2+\Big(\frac{4}{3}-\frac{\gamma}{12}\Big)y_3-\frac{1}{12}y_4\Big]\\
y_3'' &= 100\Big[-\frac{1}{12}y_1+\frac{4}{3}y_2-\frac{5}{2}y_3+\frac{4}{3}y_4\Big]\\
y_4'' &= 100\Big[-\frac{1}{2}y_1+\frac{7}{6}y_2-\Big(\frac{1}{3}-\frac{\gamma}{12}\Big)y_3-\frac{5}{4}y_4\Big]
\end{aligned}\right\} \quad \cdots\cdots\cdots \text{(v)}.$$

Putting $s=1, 2, 3, 4$ in (iv) and substituting (v), we get four simultaneous equations with respect to y_1, y_2, y_3, y_4 and the simplified forms of them are

$$\left.\begin{aligned}
-122.69792y_1-36.61458y_2+325.98876y_3-54.86979y_4&=0.3323007\\
134.34896y_1-251.43056y_2+116.36127y_3-11.92708y_4&=0.3317841\\
-7.60417y_1+131.75347y_2-246.59288y_3+131.01563y_4&=0.3312675\\
-49.55729y_1+115.93750y_2-11.54703y_3-125.04167y_4&=0.3307509
\end{aligned}\right\} \quad \cdots\cdots \text{(vi)}$$

Table. 2.

s	y_s	Y_s
0	0.9794861	0.97943
1	0.7894676	0.78942
2	0.5955576	0.59552
3	0.3986945	0.39867
4	0.1998461	0.19983
5	0.0000000	0.00000

These are linear with respect to unknowns and we can easily solve them and the result is shown in the table. 2, where y_0 is calculated from $y_0=\gamma y_3$.

Next, we consider the errors of above solutions. For the expression of $F_{5.5}$, by the equation (10), the error $R_{5.5}$ satisfies

$$|R_{5.5}|<\frac{h^7}{6!}\Big[\sum_{i=1}^{5}\Big|\frac{1}{h}A_{i.5}\cdot i^6+\frac{5^7}{7}\Big]G\fallingdotseq 0.0000029\,G$$

similarly for y_5',

$$|R_{5.5}'<0.00015\,G$$

for y_1'', y_2'', y_3'', y_4''

$$|R_{5.2}''| \text{ and } |R_{5.3}''|<0.00027\,G$$
$$|R_{5.1}''| \text{ and } |R_{5.5}''<0.00059\,G$$

where G, G are the upper bounds of $|d^6[2(x+\xi)y(\xi)]/d\xi^6|$, $|d^6y(x)/d^6x|$ respectively.

Table. 3.

s	y_s''	y_s'''
0	− 0.47951	0.878
1	− 0.38948	0.932
2	− 0.29554	0.955
3	− 0.19870	0.980
4	− 0.09986	0.992
5	0.00001	1.000

By substituting the numerical values of y_s (which were calculated) into the approximate formulas, we can find the values of y_s'' and then the values of y_s''' from $y_s''' = \sum_{i=0}^{p} B_{i,j}\, y_{i+s-j}''$ and the results of my calclation is shown in the table. 3.

Next, differentiating (iii) with respect to x gives

$$y^{(n+2)}(x) + y^{(n)}(x) = 0, \quad \text{when } n \geqq 2$$

$$\therefore \quad y^{(v)}(x) = -y'''(x), \quad y^{(vi)}(x) = -y^{(iv)}(x) = y''(x)$$

hence

$$|y^{(vi)}(x)| = |y''(x)| < 0.48, \quad |y^{(v)}(x)| = |y'''(x)| = 1,$$

while

$$d^6[2(x+\xi)y(\xi)]/d\xi^6 = 2(x+\xi)y^{(vi)}(\xi) + 12y^{(v)}(\xi)$$

therefore

$$|[2(x+\xi)y(\xi)]^{(vi)}| < 2(0.5+0.5)\times 0.48 + 12\times 1 < 13.$$

Accordingly we can put to $G = 13$, $G = 0.48$ and get

$$|R_{5.5}| < 0.0000029\times 13 \fallingdotseq 0.000038$$

$$|R_{5.5}'| < 0.00015\times 0.48 \fallingdotseq 0.000072$$

$$|R_{5.1}'| \text{ and } |R_{5.5}''| < 0.00059\times 0.48 \fallingdotseq 0.00028$$

$$|R_{5.2}''| \text{ and } |R_{5.3}''| < 0.00027\times 0.48 \fallingdotseq 0.00013.$$

Now, let the errors of the approximate equations (vi) be r_1, r_2, r_3, r_4 respectively and the errors of y_1, y_2, y_3, y_4 which are obtained from (vi) be ε_1, ε_2, ε_3, ε_4 respectively, then we have following accurate equations

$$\left.\begin{aligned} -122.69792\,(y_1+\varepsilon_1) - 36.61458(y_2+\varepsilon_2) + 325.98876(y_3+\varepsilon_3) - 54.86979(y_4+\varepsilon_4) &= 0.3323007 + r_1 \\ 134.34896\,(y_1+\varepsilon_1) - 251.43056(y_2+\varepsilon_2) + 116.36127(y_3+\varepsilon_3) - 11.92708(y_4+\varepsilon_4) &= 0.3317841 + r_2 \\ -\ 7.604167(y_1+\varepsilon_1) + 131.75347(y_2+\varepsilon_2) - 246.59288(y_3+\varepsilon_3) + 131.01563(y_4+\varepsilon_4) &= 0.3312675 + r_3 \\ -\ 49.55729\,(y_1+\varepsilon_1) + 115.93750(y_2+\varepsilon_2) - 11.54703(y_3+\varepsilon_3) - 125.04167(y_4+\varepsilon_4) &= 0.3307509 + r_4 \end{aligned}\right\} \cdots \text{(vii)}.$$

Subtracting (vi) from (vii), we get the simultaneous equations with respect to ε_1, ε_2, ε_3, ε_4 and solving them we obtain

$$\varepsilon_1 \fallingdotseq \frac{1}{122}(48\,r_1 + 76\,r_2 + 96\,r_3 + 71\,r_4)$$

$$\varepsilon_2 \fallingdotseq \frac{1}{122}(36\,r_1 + 58\,r_2 + 72\,r_3 + 54\,r_4)$$

$$\varepsilon_3 \fallingdotseq \frac{1}{122}(24\,r_1 + 39\,r_2 + 48\,r_3 + 37\,r_4)$$

$$\varepsilon_4 \fallingdotseq \frac{1}{122}(12\,r_1 + 19\,r_2 + 24\,r_3 + 18\,r_4)\,.$$

While

$$|r_1| \leqq |R_{5.5}| + |R'_{5.5}| + |R''_{5.1}| < 0.00039$$

similarly

$$|r_2| < 0.00024, \quad |r_3| < 0.00024, \quad |r_4| < 0,00039,$$

hence

$$|\varepsilon_1| < \frac{1}{122}(48\times 39 + 76\times 24 + 96\times 24 + 71\times 39)\times 10^{-5} \fallingdotseq 0.00072,$$

similarly

$$|\varepsilon_2| < 0.00054, \quad |\varepsilon_3| < 0.00037, \quad |\varepsilon_4| < 0.00018,$$

and

$$|\varepsilon_0| = \gamma|\varepsilon_3| < 0.00091.$$

Therefore the errors of the values of y_s in the Table. 2. are within above bounds. One will find that $y(x) = 0.5 - x + sin\,(0.5 - x)$ is the solution of this problem by substitution and Y_s shown in the Table. 2. are the values calculated from this formula and y_s, Y_s coinside within above bounds.

Example 3. Solve the equation

$$y''(x) + \frac{1}{64y(x)^3} + \sqrt{1.024}\,\{y(x)^2 - 1\} - \frac{1}{\sqrt{1.4}}(0.81x - 1.85) + \int_0^1 \{(x-6\xi)y(\xi) + \xi^2(x+\xi)y''(\xi)\}d\xi = 0 \quad \cdots\cdots \text{(i)}$$

$$0 \leqq x \leqq 1$$

under the conditions

$$y(0) : y(0.4) : y(0.8) = 1 : 1 : 3 \quad \cdots\cdots \text{(ii)}.$$

Solution :-

At first, as the integrand is linear form, (i) becomes

$$y''(x) + \frac{1}{64y(x)^3} + \sqrt{1.024}\,\{y(x)^2 - 1\} - \frac{1}{\sqrt{1.4}}(0.81x - 1.85) + (x+1)y'(1) - (2x+3)y(1) + 3x\int_0^1 y(\xi)d\xi = 0 \quad \cdots\cdots \text{(iii)}.$$

Put $a=0$, $b=1$, $m=10$ and $h=0.1$, $x_i = \xi_i = 0.1\ i$ and

$$I = \int_0^1 y(\xi)d\xi = \int_0^{0.2} + \int_{0.2}^{0.6} + \int_{0.6}^{1.0}$$

and by applying approximate formulas for definite integral, that is[7],

$p=5$ and F_2 for $\int_0^{0.2}$, $p=4$ and F_4 for $\int_{0.2}^{0.6}$ and $\int_{0.6}^{1.0}$, we get

$$I = \frac{h}{90}(28y + 12y + 12y + 112y + 12y + 12y + 56y + 12y + 28y + 12y + 28y) \quad \text{(iv)}$$

and

$$y''(x)+\frac{1}{64y(x)^3}+\sqrt{1.024}\{y(x)^2-1\}-\frac{1}{\sqrt{1.4}}(0.81x-1.85)$$
$$+(x+1)y'_0-(2x+3)y_0+3xI=0 \quad \cdots\cdots \text{(v)}.$$

Putting $x=x_s$ in (v) gives

$$y''_s+\frac{1}{64y_s^3}+\sqrt{1.024}\,(y_s^2-1)-\frac{1}{\sqrt{1.4}}(0.081s-1.85)$$
$$+(0.1s+1)y'_{10}-(0.2s+3)y_{10}+0.3sI=0 \quad \cdots\cdots \text{(vi)}.$$

While by the approximate formulas for differential coefficients and substituting $y=\sqrt{2}\,y$, $y_s=\sqrt{3}\,y_0$ we have ($p=6$)

$$\left.\begin{aligned}
y''_1&=100\left[\left(\frac{137}{180}-\frac{19}{12}\sqrt{2}\right)y_0-\frac{49}{60}y_1-\frac{17}{12}y_2+\frac{47}{18}y_3+\frac{31}{60}y_5-\frac{13}{180}y_6\right]\\
y''_2&=100\left[\left(-\frac{13}{180}+\frac{\sqrt{2}}{12}\right)y_0+\frac{19}{15}y_1-\frac{7}{3}y_2+\frac{10}{9}y_3-\frac{1}{15}y_5+\frac{1}{90}y_6\right]\\
y''_3&=100\left[\left(\frac{1}{90}+\frac{3}{2}\sqrt{2}\right)y_0-\frac{3}{20}y_1+\frac{3}{2}y_2-\frac{49}{18}y_3-\frac{3}{20}y_5+\frac{1}{90}y_6\right]\\
y''_4&=100\left[-\frac{49}{18}\sqrt{2}\,y_0+\frac{1}{90}y_1-\frac{3}{20}y_2+\frac{3}{2}y_3+\frac{3}{2}y_5-\frac{3}{20}y_6-\frac{1}{90}y_7\right]\\
y''_5&=100\left[\left(\frac{3\sqrt{2}}{2}+\frac{\sqrt{3}}{90}\right)y_0+\frac{1}{90}y_2-\frac{3}{20}y_3-\frac{49}{18}y_5+\frac{3}{2}y_6-\frac{3}{20}y_7\right]\\
y''_6&=100\left[\left(-\frac{3\sqrt{2}}{20}-\frac{3\sqrt{3}}{20}\right)y_0+\frac{1}{90}y_3+\frac{3}{2}y_5-\frac{49}{18}y_6+\frac{3}{2}y_7+\frac{1}{90}y_9\right]\\
y''_7&=100\left[\left(\frac{1}{90}+\frac{3\sqrt{3}}{2}\right)y_0-\frac{3}{20}y_2+\frac{3}{2}y_5-\frac{49}{18}y_7-\frac{3}{20}y_9+\frac{1}{90}y_{10}\right]\\
y''_8&=100\left[\left(\frac{\sqrt{2}}{90}-\frac{7\sqrt{3}}{3}\right)y_0-\frac{1}{15}y_5+\frac{1}{12}y_6+\frac{10}{9}y_7+\frac{19}{15}y_9-\frac{13}{180}y_{10}\right]\\
y''_9&=100\left[\left(-\frac{13\sqrt{2}}{180}-\frac{17\sqrt{3}}{12}\right)y_0+\frac{31}{60}y_5-\frac{19}{12}y_6+\frac{47}{18}y_7-\frac{49}{60}y_9+\frac{137}{180}y_{10}\right]
\end{aligned}\right\} \quad \cdots\cdots \text{(vii)}$$

and

$$y''_{10}=10\left[\left(\frac{\sqrt{2}}{6}+\frac{15\sqrt{3}}{2}\right)y_0-\frac{6}{5}y_5+\frac{15}{4}y_6-\frac{20}{3}y_7-6y_9+\frac{49}{20}y_{10}\right]$$

By putting $s=1, 2, 3, \cdots\cdots, 9$ in (vi) and substituting (vii), we obtain nine simultaneous equations with respect to y_0, y_1, y_2, y_3, y_5, y_6, y_7, y_9, y_{10}, and the simplified forms of these are as follows.

$$
\left.\begin{array}{l}
-\ 2.27067y_0 - 81.62367y_1 - 141.65267y_2 + 261.15844y_3 + 38.50967y_5 + 34.04645y_6 - 73.29066y_7 - 65.85733y_8 + 23.75933y_{10} = A_1 \\
163.37278y_0 + 126.75267y_1 - 233.30533y_2 + 111.20578y_3 - 20.98067y_5 + 46.14844y_6 - 79.91467y_7 - 71.91467y_8 + 26.01867y_{10} = A_2 \\
\cdots \\
-199.62141y_0 + 1.28311y_1 - 14.94400y_2 + 150.18933y_3 + 193.37200y_5 + 37.57467y_6 - 92.05155y_7 - 83.82933y_8 + 30.58733y_{10} = A_4 \\
\cdots \\
164.71404y_0 + 0.25800y_1 + 0.08400y_2 + 1.39511y_3 + 131.05800y_5 - 212.11023y_6 + 43.58933y_7 - 94.63289y_8 + 35.05600y_{10} = A_6 \\
486.56151y_0 + 0.30100y_1 + 0.09800y_2 + 0.33133y_3 - 35.09900y_5 + 213.88067y_6 - 385.25689y_7 - 116.70133y_8 + 38.42644y_{10} = A_7 \\
-164.11678y_0 + 0.34400y_1 + 0.11200y_2 + 0.37867y_3 - 27.92267y_5 + 75.98266y_6 - 8.54756y_7 + 19.00800y_8 + 32.35245y_{10} = A_8 \\
-\ 3.85604y_0 + 0.38700y_1 + 0.12600y_2 + 0.42600y_3 + 29.25367y_5 - 86.91533y_6 + 134.82844y_7 - 195.28267y_8 + 117.94511y_{10} = A_9
\end{array}\right\} \quad \cdots \quad \text{(viii)}
$$

Table. 4

s	$y_{s,1}$	$A_{s,1}$	$y_{s,2}$	$A_{s,2}$	$y_{s,3}$	$A_{s,3}$	$y_{s,4}$	$A_{s,4}$	$y_{s,5}$	$A_{s,5}$	$y_{s,6}$	Y_s
0	0.5		0.3126		0.31564		0.31514		0.31504		0.31506	0.316228
1	0.5	−1.772	0.3488	−1.88520	0.35307	−1.87505	0.35243	−1.87653	0.35233	−1.87674	0.35235	0.353553
2	0.5	−1.703	0.3822	−1.75311	0.38676	−1.74687	0.38606	−1.74780	0.38597	−1.74792	0.38599	0.387298
3	0.5	−1.635	0.4132	−1.65122	0.41766	−1.64795	0.41695	−1.64845	0.41684	−1.64853	0.41686	0.418330
4	0.5	−1.566	0.4421	−1.56712	0.44638	−1.56582	0.44568	−1.56602	0.44553	−1.56606	0.44556	0.447214
5	0.5	−1.498	0.4690	−1.49410	0.47335	−1.49341	0.47267	−1.49410	0.47248	−1.49410	0.47250	0.474342
6	0.5	−1.429	0.4944	−1.42824	0.49890	−1.42930	0.49818	−1.42912	0.49800	−1.42907	0.49802	0.500000
7	0.5	−1.361	0.5184	−1.36724	0.52328	−1.36928	0.52251	−1.36894	0.52233	−1.36887	0.52235	0.524405
8	0.5	−1.292	0.5414	−1.30975	0.54670	−1.31275	0.54585	−1.31226	0.54566	−1.31216	0.54569	0.547723
9	0.5	−1.224	0.5636	−1.25494	0.56931	−1.25889	0.56836	−1.25821	0.56817	−1.25808	0.56820	0.570088
10	0.5		0.5854		0.59127		0.59016		0.58998		0.59001	0.591608

$y_{1,n}$, $y_{8,n}$ were calculated from the value of $y_{0,n}$ by the relations $y_{1,n}=\sqrt{2}\ y_{0,n}$, $y_{8,n}=\sqrt{3}\ y_{0,n}$.

where

$$\left.\begin{aligned} C_s &= \frac{1}{64y_s^2} + 1.01192885y_s^2 \\ A_s &= 0.0684574945 - 1.462342478 - C_s \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots \text{(ix)}.$$

$$s = 1, 2, 3, \cdots\cdots\cdots, 9$$

These equations are not linear and it is very difficult to solve them exactly. But for these equations, an iterative approximation method is very effective, that is as follows. At first, take any values $y_{0.1}$, $y_{1.1}$, $y_{2.1}$, $y_{3.1}$, $y_{5.1}$, $y_{6.1}$, $y_{7.1}$, $y_{9.1}$, $y_{10.1}$ as the first approximate values of y_0, y_1, y_2, y_3, y_5, y_6, y_7, y_9, y_{10} respectively and calculate the second approximate values, the third approximate values, ····· and so on by the relations that

in the equations (viii), put $y_{0.n}$, $y_{1.n}$, $y_{2.n}$, ·····, $y_{10.n}$ in places of y_0, y_1, y_2, ·····, y_{10} in the left sides respectively and $A_{1.n-1}$, $A_{2\,n-1}$, ·····, $A_{9.n-1}$ in places of A_1, A_2, ·····, A_9 in the right sides respectively, where

$$C_{s.n} = \frac{1}{64y_{s.n}^2} + 1.01192885y_{s.n}^2$$

$$A_{s.n} = 0.0684574945 - 1.462342478 - C_{s.n}$$

and give the values of $A_{1.n-1}$, $A_{2.n-1}$, ·····, $A_{9.n-1}$ which are calculated from the values of $y_{1.n-1}$, $y_{2.n-1}$, ·····, $y_{9.n-1}$ and then solve these equations with respect to $y_{0.n}$, $y_{1.n}$, $y_{2.n}$, ·····, $y_{10.n}$.

If $y_{s.n} \equiv y_{s.n+1}$ is reached, then $y_{s.n} = y_s$ are required solutions. The result of my calculation is shown in the table. 4.

The errors of these solutions may be estimated similarly as in the example. 2, but the procedure is so troublesome that I have no leisure to do it. As a matter of fact $y(x) = \frac{1}{2}\sqrt{x+0.4}$ is the solution of this problem and Y_s in the table. 4 are the values calculated from this formula, and they show that y_s, Y_s coinside within about 0.3 %.

6. Remarks.

In above discussion, we considered the interval of x to be from a to b but it is obvious that we can expand this interval to both sides at any finite amount.

If $\varphi(x, \xi, y(\xi), y'(\xi), y''(\xi))$ involves none of y, y', y'' the equation (1) becomes an ordinary differential equation, and if $f(x, y(x), y'(x), y''(x))$, $\varphi(x, \xi, y(\xi), y'(\xi), y''(\xi))$ both involve none of y', y'' the equation (1) becomes a pure integral equation, hence above method is applicable to differential equations and integral equations.

References

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ASCENSION OF A SMALL MASS OF AIR AND THE CHANGES IN ITS STATE (I) DRY AIR

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(Received February 28, 1956)

The ascending motion of air and the changes in its state are complex and it is very difficult to solve the problem theoretically. We have attempted here to solve the problem approximately under various simple assumptions and have obtained some results.

1. Introduction

Let us suppose that there is a small dry mass of air in the atmosphere at rest, the temperature of the mass of air is higher than that of the surrounding atmosphere. The mass of air will move upward and its state will be changed slowly by the change in the boundary condition which are caused by the upward motion of the mass of air.

As to the changes in the state of gas produced by the changes in the boundary conditions, we have discussed various cases in the previous paper[1], and so we can imagine the changes in the state of the mass of air produced by the change in the surrounding atmosphere which is caused by the ascending motion.

Ascending mass of air will be subjected to some complex resistance from surrounding atmosphere, but, it is treated here two cases, that is, in one case there is no resistance, and in another case there is a resistance proportional to the velocity of the mass of air. Practically, the atmosphere is always in motion and changes its state in some complex manner. Especially, it is considered that by the motion of the mass of air, the states of the surrounding atmosphere and the mass of air itself will be disturbed complicatedly, a part of the mass of air will be mixed with the surrounding atmosphere and the form of the mass of air will be changed considerably. Thus the resistance to the motion of the mass of air is considered very complex. But if we take in consideration such circumstances, the problem becomes very difficult, or even almost insoluble. It is assumed, therefore, that the atmosphere is at rest and keeps steady state, the temperature of the atmosphere changes with height in a constant lapse rate, and the pressure of the atmosphere changes with height corresponding to the changes of the temperature, and also assumed that the pressure of the mass of air is always equal to that of the surrounding atmosphere, as the mass of air is small. It is assumed, more over, that in one case, there is no heat exchange between the mass of air and the surrounding atmosphere, in another case, the heat transfers between them in proportion to the difference of the temperatures of the mass of air and its surrounding atmosphere.

2. Relation between Temperature and Height in the Atmosphere

If the pressure, the temperature and the density are p, T_1 and ρ respectively at height h in the atmosphere, there are relations

$$dp=-dh\rho g, \quad \frac{p}{\rho}=R'T_1,$$

where R' is the specific gas constant of air. From them it is obtained

$$d\log p=-\frac{g}{RT}dh. \quad \cdots\cdots (1)$$

If the atmosphere is in the adiabatic equilibrium state, there is the relation $pT_1^{-\frac{C_p}{R'}}=const.$ where C_p is the specific heat at constant pressure. Putting this relation in (1), we get

$$\frac{dT_1}{dh}=-\frac{g}{C_p}, \quad T_1=T_{10}-\frac{g}{C_p}h, \quad \cdots\cdots (2)$$

where T_{10} is the value of T_1 at $h=0$.

$\frac{g}{C_p}$ is the dry adiabatic lapse rate and is expressed by Γ. Actual lapse rate γ is in most cases small compared to Γ, and changes with time and place. It is assumed, for simplicity, that the value of γ is constant in the atmosphere, then the temperature T_1 of the atmosphere at height h is expressed by

$$T_1=T_{10}-\gamma h. \quad \cdots\cdots (3)$$

3. Relation between Pressure and Height in the Atmosphere

Pressure p changes with height h by the relation (1), and temperature T_1 changes with height h by the relation (3). Putting (3) in (1), we have

$$d\log p=-\frac{g}{R'}\frac{dh}{(T_{10}-\gamma h)}. \quad \cdots\cdots (4)$$

Integrating this and putting the condition that the pressure at $h=0$ is p_0, we get

$$\frac{p}{p_0}=\left(\frac{T}{T_{10}}\right)^{\frac{g}{R'\gamma}}=\left(1-\frac{\gamma h}{T_{10}}\right)^{\frac{g}{R'\gamma}}, \quad \cdots\cdots (5)$$

or

$$pT^{\frac{g}{R'\gamma}}=p_0T_{10}^{\frac{g}{R'\gamma}}. \quad \cdots\cdots (6)$$

This is the relation which expresses change in pressure with height.

If the relation $\frac{\gamma h}{T_{10}}\ll 1$ is satisfied, (5) becomes

$$p=p_0\left(1-\frac{gh}{RT}\right). \quad \cdots\cdots (7)$$

which shows that pressure p changes independently of the value of γ. If the value of γh can not be neglected compared to T_{10}, the pressure p changes a little with the value of γ. To know the effect of γ, we differentiate $\log p$ with γ using (5).

$$\frac{d\log p}{d\gamma} = -\frac{g}{R'\gamma^2}\log\left(1-\frac{\gamma h}{T_{10}}\right)+\frac{g}{R'\gamma}\left(-\frac{h}{T_{10}-\gamma h}\right)$$

$$= -\frac{g}{R'\gamma}\left\{\frac{\gamma h}{T_{10}-\gamma h}+\log\left(1-\frac{\gamma h}{T_{10}}\right)\right\}.$$

As we can use the expansions

$$\frac{\gamma h}{T_{10}-\gamma h}=\frac{\gamma h}{T_{10}}\left\{1+\frac{\gamma h}{T_{10}}+\left(\frac{\gamma h}{T_{10}}\right)^2+\cdots\cdots\right\},$$

and

$$\log\left(1-\frac{\gamma h}{T_{10}}\right)=-\frac{\gamma h}{T_{10}}-\frac{1}{2}\left(\frac{\gamma h}{T_{10}}\right)^2-\cdots\cdots,$$

we know that $\frac{d\log p}{d\gamma}<0$ which means p decreases with the increase of the value of the value of γ.

4. Change in Energy of the Ascending Mass of Air when there is no Resistance

It is assumed that a small mass of air has a distinct boundary with the surrounding atmosphere, and the temperature T of the mass of air is different from the temperature of the surrounding atmosphere, the pressure of the mass of air is equal to that of the surrounding atmosphere.

If the volume of unit mass of the air is v, there is the relation $pv = R'T$, and if the mass of the volume v of the surrounding atmosphere is m, there is the relation $pv = mR'T_1$.

The ascending force, acting on unit mass of the air, which is equal to the difference of the buoyancy and the weight of the air, is expressed by

$$(m-1)g=\frac{T-T_1}{T_1}g=\frac{\triangle T}{T_1}g,\quad \triangle T=T-T_1. \qquad \cdots\cdots\cdots\cdots\cdots (8)$$

If the air ascends dh in time dt, the work which the air accepts per unit mass is expressed by

$$dw=\frac{\triangle T}{T_1}gdh=-R'\triangle T d\log p^{(2)}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots (9)$$

It is assumed that the mass of air accepts heat from the surrounding atmosphere in the form $k(T_1-T)$ per unit mass, where k is a constant. If the volume and the temperature of unit mass of air increases dv and dT in time dt, we can express the work done by unit mass, the

increase of the internal energy and the increase of the kinetic energy of unit mass per unit time by $p\frac{dv}{dt}$, $C\frac{dT}{dt}$ and $\frac{1}{2}\frac{d}{dt}\left(\frac{dh}{dt}\right)^2$ respectively, where C_v is the specific heat at constant volume. If there is no resistance, there exists the following relation.

$$-R'\triangle T\frac{d\log p}{dt}+k(T_1-T)=C_v\frac{dT}{dt}+p\frac{dv}{dt}+\frac{1}{2}\frac{d}{dt}\left(\frac{dh}{dt}\right)^2. \quad \cdots\cdots \quad (10)$$

Now, as there are the relations,

$$C_v dT + pdv = C_p dT - vdp = C_p dT - R'Td\log p,$$

$$\frac{d\log p}{dt} = -\frac{g}{R'T_1}\frac{dh}{dt}, \qquad \frac{T}{T_1}g = -\frac{d^2h}{dt^2},$$

and

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dh}{dt}\right)^2=\frac{dh}{dt}\frac{d^2h}{dt^2},$$

(10) becomes

$$-k\triangle T=C_p\frac{dT}{dt}-R'T_1\frac{d\log p}{dt}+\frac{\triangle T}{T_1}g\frac{dh}{dt}. \quad \cdots\cdots \quad (11)$$

Using (1), we get from (11) the following expression

$$-k\triangle T=C_p\frac{dT}{dt}+\frac{T}{T_1}g\frac{dh}{dt}. \quad \cdots\cdots \quad (12)$$

5. Changes in Temperature of the Ascending Mass of Air

Temperature of the mass of air changes by the changes in its boundary conditions, caused by its ascension and by the transference of heat between the mass of air and its surrounding atmosphere. We shall consider two cases where there is no transference of heat, and there is transference of heat in proportion to the temperature difference.

(i). The case of $k=0$.

In this case we get from (12)

$$\frac{dT}{dh}=-\frac{g}{C_p}\frac{T}{T_1}=-\Gamma\frac{T}{T}=-\Gamma\frac{T}{T_0+\Gamma h}. \quad \cdots\cdots \quad (13)$$

Integrating this and putting the condition that at $h=0$, $T=T_0$, we get

$$\frac{T}{T_0}=\left(\frac{T_1}{T_{10}}\right)^{\frac{\Gamma}{\gamma}}=\left(1-\frac{\gamma h}{T_{10}}\right)^{\frac{\Gamma}{\gamma}}. \quad \cdots\cdots \quad (14)$$

This shows the relation of temperature and height of the mass of air when there is no heat transference. Table 1. shows the relation of temperature T and height h of the mass of air which is obtained from (14) by giving various values to T_{10} and T_0 and assuming $\frac{\Gamma}{\gamma}=2$.

Table 1. Relation between height h and temperature T. $\left(k=0,\ \frac{\Gamma}{\gamma}=2\right)$

h	0^{km}	0.4	0.8	1.2	1.6	2.0	2.4	2.8	3.2	3.6	4.0	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7.6	8.0
T	260	258	256	254	252	250	248	246	244	242	240	238	236	234	232	230	228	226	224	222	220
T	265	261	257	253	249	245															
T	270	266	262	258	254	250	246	242	238	234	230										
T	280	276	272	267	263	259	255	251	247	243	239	235	231	227	223	219	215	212	208	204	
T	270	268	266	264	262	260	258	256	254	252	250	248	246	244	242	240	238	236	234	232	230
T	275	271	267	263	259	255															
T	280	276	272	268	264	260	256	252	248	244	240										
T	285	281	277	272	268	264	260	256	252	248	244	240	237	233	229	225					
T	290	286	282	277	273	269	265	261	257	253	249	245	241	237	233	229	225	222	218	214	210
T	280	278	276	274	272	270	268	266	264	262	260	258	256	254	252	250	248	246	244	242	240
T	285	281	277	273	269	265															
T	290	286	282	278	274	270	266	262	258	254	250										
T	295	291	287	282	278	274	270	266	262	258	254	250	247	243	239	235					
T	300	296	292	287	283	279	275	271	267	263	259	255	251	247	243	239	235	232	228	224	220
T	290	288	286	284	282	280	278	276	274	272	270	268	266	264	262	260					
T	295	291	287	283	279	275															
T	300	296	292	288	284	280	276	272	268	264	260										
T	305	301	297	293	288	284	280	276	272	268	264	260	257	253	249	245					
T	300	298	296	294	292	290	288	286	284	282	280	278	276	274	272	270					
T	305	301	297	293	289	285															
T	310	306	302	298	294	290	286	282	278	274	270										
T	315	311	307	303	298	294	290	286	282	278	274	271	267	263	259	255					
T	310	308	306	304	302	300	298	296	294	292	290	288	286	284	282	280					
T	315	311	307	303	299	295															
T	320	316	312	308	304	300	296	292	288	284	280										
T	325	321	317	313	308	304	300	296	292	288	284	281	277	273	269	265					
T	320	318	316	314	312	310	308	306	304	302	300	298	296	294	292	290					
T	325	321	317	313	309	305															
T	330	326	322	318	314	310	306	302	298	294	290										

From (5) and (14) we obtain the relation of pressure and temperature as the following.

$$\frac{p}{p_0}=\left(\frac{T}{T_0}\right)^{\frac{\gamma}{\Gamma}\frac{g}{R\gamma}}=\left(\frac{T}{T_0}\right)^{\frac{C_p}{R}}. \qquad \cdots\cdots (15)$$

This is just the relation which expresses the adiabatic change.

We can obtain the height h_1 at which T becomes equal to T_1, by putting $T=T_1$ in the expression (14), thus

$$h_1=\frac{1}{\gamma}\left\{T_{10}-\left(\frac{T_{10}^{\Gamma}}{T_0^{\gamma}}\right)^{\frac{1}{\Gamma-\gamma}}\right\}=\frac{T_{10}}{\gamma}\left\{1-\left(\frac{T_{10}}{T_0}\right)^{\frac{\gamma}{\Gamma-\gamma}}\right\}. \qquad \cdots\cdots (16)$$

In a special case where $\frac{\Gamma}{\gamma}=2$ is satisfied, h_1 becomes

$$h_1=\frac{T_{10}-T_0}{\gamma T_0}. \qquad \cdots\cdots (16')$$

By (13), we know that $-\frac{dT}{dh}$ is equal to Γ at the place where $T=T_0$, $-\frac{dT}{dh}$ is greater than Γ at the place where $T>T_0$, $-\frac{dT}{dh}$ is smaller than Γ at the place where $T<T_0$.

(ii) The case of $k\neq 0$.

From (12) we have

$$-\frac{k\triangle T dt}{T}=C_p\frac{dT}{T}+g\frac{dh}{T_{10}-\gamma h}, \qquad \cdots\cdots (17)$$

and by integration

$$\triangle S=C_p\left(log\frac{T}{T_0}-\frac{\Gamma}{\gamma}log\frac{T_1}{T_{10}}\right), \qquad \cdots\cdots (18)$$

where

$$\triangle S=-\int\frac{k\triangle T dt}{T},$$

which is nearly the change in entropy of the mass of air. From (18) we get

$$\frac{T}{T_0}=\left(\frac{T_1}{T_{10}}\right)^{\frac{1}{\gamma}}e^{\frac{\triangle S}{C_p}}. \qquad \cdots\cdots (19)$$

If $\triangle S=0$, (19) becomes (14). We know also from (19) that if $\triangle S>0$ or $T_1>T$, T is higher than the temperature of the adiabatic change, and if $\triangle S<0$ or $T_1<T$, T is lower than the temperature of the adiabatic change. Further from (19) we have

$$\frac{dT}{dh}=-\Gamma\frac{T}{T_1}+\frac{T}{C_p}\frac{d(\triangle S)}{dh}. \qquad \cdots\cdots (20)$$

where $T\frac{d(\triangle S)}{dh}$ is the heat accepted by the mass of air while it ascends unit height, and the value of $T\frac{d(\triangle)S}{dh}$ is supposed to be generally small. From (20) we know that if

$T = T_1$, $\frac{d(\triangle S)}{dh} = 0$ and $-\frac{dT}{dh} = \Gamma$. We know also that if $T > T_1$, $\frac{d(\triangle S)}{dh} < 0$ and $-\frac{dT}{dh} > \Gamma\frac{T}{T_1}$, and if $T < T_1$, $\frac{d(\triangle S)}{dh} > 0$ and $-\frac{dT}{dh} < \Gamma\frac{T}{T_1}$.

The height h_1, at which T becomes equal to T_1, is obtained from (19) by putting $T = T_1$ in it.

$$h_1 = \frac{T_{10}}{\gamma}\left\{1 - \left(\frac{T_{10}}{T_0}e^{-\frac{\triangle S}{C_p}}\right)^{\frac{\gamma}{\Gamma-\gamma}}\right\}. \quad \cdots\cdots (21)$$

By combining (5) and (19) we get the relation between pressure and temperature of the mass of air as follows.

$$\frac{T}{T} = \left(\frac{p}{p_0}\right)^{\frac{R'}{C_p}} e^{\frac{\triangle S}{C_p}}. \quad \cdots\cdots (22)$$

If $\triangle S = 0$, (22) becomes (15).

6. Change in Energy of the Ascending Mass of Air when there is a Resistance

As previously stated, the problem of the resistance to the motion of air is extremely complicated and very difficult to solve theoretically. Here, for simplicity, it is assumed that the resistance to the motion of the air is proportional to the velocity of it, and is expressed in the form $-K\frac{dh}{dt}$, where K is a constant. As the work done by the mass of air against the resistance per unit time is $K\left(\frac{dh}{dt}\right)^2$, the energy equation becomes

$$-R'\triangle T\frac{d\log p}{dt} + k(T_1 - T) = C_v\frac{dT}{dt} + p\frac{dv}{dt} + \frac{1}{2}\frac{d}{dt}\left(\frac{dh}{dt}\right)^2 + K\left(\frac{dh}{dt}\right)^2. \quad (23)$$

Putting the relations

$$C_v dT + pdv = C_p dT - R'Td\log p,$$

$$\frac{1}{2}\frac{d}{dt}\left(\frac{dh}{dt}\right)^2 = \frac{dh}{dt}\frac{d^2h}{dt^2}, \quad \frac{d\log p}{dt} = -\frac{g}{R'T_1}\frac{dh}{dt},$$

and

$$\frac{\triangle T}{T_1}g - K\frac{dh}{dt} = \frac{d^2h}{dt^2}$$

in (23), we get

$$-k\triangle T = C_p\frac{dT}{dt} + \frac{T}{T_1}g\frac{dh}{dt}, \quad \cdots\cdots (24)$$

which has the same form as (12). In this case, therefore, the results obtained in **5.** can be applied. Thus, there are the relations

$$\frac{T}{T} = \left(\frac{T_1}{T_{10}}\right)^{\frac{\Gamma}{\gamma}} e^{\frac{\triangle S}{C_p}},$$

and

$$\frac{dT}{dh} = -\Gamma\frac{T}{T_1} + \frac{T}{C_p}\frac{d(\triangle S)}{dh},$$

between temperature and height of the mass of air. The height h_1 at which T becomes equal to T_1 is also expressed by

$$h_1 = \frac{T}{\gamma}\left\{1 - \left(\frac{T}{T}\,e^{-\frac{\triangle S}{C_p}}\right)^{\frac{\Gamma}{\gamma}}\right\}.$$

7. Relation between Height and Ascending Velocity

For simplicity, we consider here a special case where $\triangle S = 0$ and $\frac{\Gamma}{\gamma} = 2$ are satisfied and there is no resistance.

In this case the relation

$$\frac{d^2h}{dt^2} = \frac{\triangle T}{T_1}g, \qquad \cdots\cdots (25)$$

is applied. From (14) we obtain $\frac{\triangle T}{T_1} = -\frac{T_0}{T_{10}^2}T_1 - 1$, hence (25) becomes

$$\frac{d^2h}{dt^2} = \left(-\frac{T_0}{T_{10}^2}T_1 - 1\right)g. \qquad \cdots\cdots (26)$$

Using the relation $T_1 = T_{10} - \gamma h$, we obtain from (26)

$$\frac{d^2T_1}{dt_2} = \gamma\left(1 - \frac{T_0}{T_{10}^2}T_1\right)g. \qquad \cdots\cdots (27)$$

Integrating this,

$$\left(\frac{dT_1}{dt}\right)^2 = 2\gamma g\left\{T_1 - \frac{1}{2}\frac{T_0}{T_{10}^2}T_1^2 + C_1\right\} \qquad \cdots\cdots (28)$$

is resulted, where C_1 is a constant which is to be determined by the initial condition. If at $t = 0$, $h = 0$ and $\frac{dh}{dt} = 0$, C_1 must have the value

$$C_1 = \frac{1}{2}T_0 - T_{10}. \qquad \cdots\cdots (29)$$

From (28) we obtain

$$\frac{2aT_1 - 1}{\sqrt{4aC_1 + 1}} = \sin\left(\sqrt{2a\gamma g}\,t + C_2\right), \qquad \cdots\cdots (30)$$

where

$$a = \frac{1}{2}\frac{T_0}{T_{10}^2}, \qquad \cdots\cdots (31)$$

and C_2 is a constant which is to be determined by the initial condition. As at $t = 0$, $T_1 = T_{10}$ is satisfied, C_2 must be determined by

$$\frac{2aT_{10}-1}{\sqrt{4aC_1+1}} = sinC_2 .$$

But we see $2aT_{10}-1 = \frac{T_0}{T_{10}} - 1 = \frac{\triangle T_0}{T_{10}}$, and

$$\sqrt{4aC_1+1} = \sqrt{4\times\frac{1}{2}\frac{T_0}{T_{10}^2}\left(\frac{1}{2}T_0 - T_{10}\right)+1} = \frac{T_0}{T_{10}} - 1 = \frac{\triangle T_0}{T_{10}},$$

so that $sinC_2 = 1$ and $sin\,(\sqrt{2a\gamma g}\,t + C_2) = cos\,\sqrt{2a\gamma g}\,t$, and therefore (30) becomes to

$$T\,T_1 - T_{10} = T_{10}\triangle T\ cos\ \sqrt{2a\gamma g}\,t,$$

which can be transformed to

$$h = \frac{T_{10}\triangle T_0}{\gamma T_0}(1 - cos\,\sqrt{2a\gamma g}\ t). \quad \cdots\cdots (32)$$

The maximum value of h is given by

$$h_m = \frac{2T_{10}\triangle T_0}{\gamma T_0} . \quad \cdots\cdots (33)$$

From (32) we have the ascending velocity

$$\frac{dh}{dt} = \frac{T_{10}}{\gamma}\frac{\triangle T_0}{T_0}\sqrt{2a\gamma g}\ sin\sqrt{2a\gamma g}\ t.$$

Putting the vaiue of $a = \frac{T_0}{2T_{10}^2}$ in this expression, we get

$$\frac{dh}{dt} = \triangle T_0\sqrt{\frac{g}{T_0\gamma}}\ sin\frac{\sqrt{T_0\gamma g}}{T_{10}}\,t. \quad \cdots\cdots (34)$$

We know that the maximum value of $\frac{dh}{dt}$ is

$$\left(\frac{dh}{dt}\right)_{max} = \triangle T_0\sqrt{\frac{g}{T_0\gamma}} . \quad \cdots\cdots (34')$$

The relation between height h and ascending velocity $\frac{dh}{dt}$ is to be obtained by eliminating t from (32) and (34), but it is obtained more easily by using (28), $T_1 = T - \gamma h$ and $C_1 = \frac{1}{2}T_0 - T_{10}$. From (28) it follows

$$\frac{dh}{dt} = \sqrt{\frac{2g}{\gamma}\left\{\frac{T_0}{2} - T_{10} + T_{10} - \gamma h - \frac{1}{2}\frac{T_0}{T_{10}^2}(T_{10}-\gamma h)^2\right\}}.$$

This becomes to

$$\frac{dh}{dt} = \frac{\sqrt{\gamma g T_0}}{T_{10}}\sqrt{h\left(\frac{2T_{10}\triangle T_0}{T_0\gamma} - h\right)}. \quad \cdots\cdots (35)$$

Table 2. shows the relation of height h and ascending velocity $\frac{dh}{dt}$, calculated by (35).

From (35) we know that $\frac{dh}{dt}$ takes the maximum value

$$\frac{\sqrt{\gamma g T_0}}{T_{10}} \cdot \frac{T_{10}\triangle T_0}{T_0\gamma} = \triangle T_0 \sqrt{\frac{g}{T_0\gamma}} ,$$

at the height $h_1 = \frac{T_{10}\triangle T_0}{T_0\gamma}$. This value of h_1 is equal to the value (16') which is obtained by putting $T = T_1$.

Table 2. Relation between height h and velocity $\frac{dh}{dt}$.

$T_{10} = 260°K$, $T_0 = 265°K$

h	$100m$	200	300	400	500	600	700	800	900	1000
$\frac{dh}{dt}$	5.95 $\frac{m}{sec}$	8.19	9.74	10.9	11.8	12.5	13.0	13.3	13.5	13.5

$T_{10} = 260$, $T_0 = 270$

h	100	200	300	500	1000	1500	2000
$\frac{dh}{dt}$	8.52	11.9	14.3	18.0	23.5	26.1	26.8

$T_{10} = 260$, $T_0 = 275$

h	100	200	500	1000	1500	2000	2500	2836
$\frac{dh}{dt}$	10.5	14.7	22.7	30.5	35.3	38.2	39.7	40.0

$T_{10} = 260$, $T_0 = 280$

h	100	300	500	1000	1500	2000	3000	3500	3714
$\frac{dh}{dt}$	11.9	20.3	26.5	35.2	42.3	46.8	51.8	52.7	52.7

$T_{10} = 270$, $T_0 = 275$

h	100	300	500	800	982
$\frac{dh}{dt}$	5.9	9.6	11.6	13.1	13.3

$T_{10} = 270$. $T_0 = 280$

h	100	300	500	1000	1500	1929
$\frac{dh}{dt}$	8.4	14.1	17.8	23.2	25.8	26.4

$T_{10} = 270$, $T_0 = 285$

h	100	500	1000	1500	2000	2500	2842
$\frac{dh}{dt}$	10.3	22.2	29.9	34.6	37.5	39.0	39.2

$T_{10} = 270$, $T_0 = 290$

h	100	500	1000	1500	2000	2500	3000	3500	3725
$\frac{dh}{dt}$	11.9	26.0	35.0	41.6	46.2	49.0	50.9	51.8	51.9

$T_{10} = 280$, $T_0 = 285$

h	100	300	500	700	983
$\frac{dh}{dt}$	5.7	9.3	11.3	12.4	13.0

$T_{10} = 280$, $T_0 = 290$

h	100	300	500	1000	1500	1931
$\frac{dh}{dt}$	8.2	13.8	17.4	22.7	25.2	25.9

$T_{10} = 280$, $T_0 = 295$

h	100	500	1000	1500	2000	2500	2848
$\frac{dh}{dt}$	10.1	21.9	29.4	34.0	36.9	38.3	38.6

$T_{10} = 280$, $T_0 = 300$

h	100	500	1000	1500	2000	2500	3000	3500	3734
$\frac{dh}{dt}$	11.7	25.5	34.8	39.9	45.2	48.2	50.0	50.9	51.0

By (35) we know that $\frac{dh}{dt} = 0$ at the maximum height

$$h_2 = \frac{2T_{10}\triangle T_0}{T_0\gamma} = 2h_1 . \qquad (36)$$

Table 3. shows the values of the maximum height h_2, calculated by (36).

Table 3. Maximum height h_2.

T_{10}	260					
T_0	265	270	275	280	285	290
h_2	1962 m	3852	5672	7428	9123	10759

T_{10}	290					
T_0	295	300	305	310	315	320
h_2	1966	3867	5706	7484	9206	10875

T_{10}	265					
T_0	270	275	280	285	290	295
h_2	1963	3855	5678	7439	9138	10780

T_{10}	295					
T_0	300	305	310	315	320	325
h_2	1967	3869	5710	7492	9219	10892

T_{10}	270					
T_0	275	280	285	290	295	300
h_2	1963	3857	5684	7449	9153	10800

T_{10}	300					
T_0	305	310	315	320	325	330
h_2	1967	3871	5714	7500	9231	10909

T_{10}	275					
T_0	280	285	290	295	300	305
h_2	1964	3860	5690	7459	9167	10820

T_{10}	305				
T_0	310	315	320	325	330
h_2	1968	3873	5719	7508	9242

T_{10}	280					
T_0	285	290	295	300	305	310
h_2	1965	3862	5696	7468	9180	10839

T_{10}	310				
T_0	315	320	325	330	
h_2	1968	3875	5723	7515	

T_{10}	285					
T_0	290	295	300	305	310	315
h_2	1965	3864	5701	7476	9193	10857

T_{10}	315		
T_0	320	325	330
h_2	1969	3877	5727

Thus the motion of the mass of air is a simple harmonic motion of amplitude h_1, the center of oscillation is the place of height h_1. The period of the motion is given by

$$\tau = \frac{2\pi}{\sqrt{2a\gamma g}} = \frac{2\pi T_{10}}{\sqrt{T_0\gamma g}}. \qquad \cdots\cdots (37)$$

Hence we can suppose that the mass of air ascends to the maximum height in time

$$\frac{\tau}{2} = \frac{\pi T_{10}}{\sqrt{T_0\gamma g}}.$$

8. Relation between Height and Ascending Velocity when there is a Resistance to the Motion

We consider here the case where $\triangle S = 0$, $\frac{\Gamma}{\gamma} = 2$ are satisfied and there act a resistance in proportion to the velocity. In this case the following equation is applied.

$$\frac{d^2h}{dt^2} = \frac{\triangle T}{T_1} g - K\frac{dh}{dt}, \qquad \cdots\cdots (38)$$

where K is a constant.

Using the relation $\frac{T}{T_0}=\left(\frac{T_1}{T_{10}}\right)^2$, we get

$$\frac{\triangle T}{T_1}g=\left(\frac{T}{T_1}-1\right)g=\left(\frac{T}{T_{10}^2}T_1-1\right)g=\left\{\frac{T_1}{T_{10}^2}(T_{10}-\gamma h)-1\right\}g.$$

Hence (38) becomes to

$$\frac{d^2h}{dt^2}+K\frac{dh}{dt}+\frac{T_0}{T_{10}^2}\gamma gh=\frac{\triangle T_0}{T_{10}}g. \quad \cdots\cdots (39)$$

The general solution of (39) is expressed by

$$h=C_1e^{-\lambda_1 t}+C_2e^{-\lambda_2 t}+\frac{T_{10}\triangle T_0}{\gamma T_0}, \quad \cdots\cdots (40)$$

where C_1 and C_2 are integration constants, and λ_1 and λ_2 are expressed by

$$\lambda_1=\frac{K}{2}-\frac{\sqrt{K^2T_{10}^2-4T_0\gamma g}}{2T_{10}} \quad \cdots\cdots (41)$$

and

$$\lambda_2=\frac{K}{2}+\frac{\sqrt{K^2T_{10}^2-4T_0\gamma g}}{2T_{10}} \quad \cdots\cdots (42)$$

respectively.

If initially $h=0$ and $\frac{dh}{dt}=0$ are satisfied, the values of C_1 and C_2 are given as

$$C_1=\frac{\lambda_2}{\lambda_1-\lambda_2}\frac{T_{10}\triangle T_0}{\gamma T_0} \text{ and } C_2=\frac{\lambda_1}{\lambda_2-\lambda_1}\frac{T_{10}\triangle T_0}{\gamma T_0}. \quad \cdots\cdots (43)$$

Thus the following solution is obtained.

$$h=\frac{T_{10}\triangle T_0}{\gamma T_0}\left\{1+\frac{\lambda_2e^{-\lambda_1 t}-\lambda_1e^{-\lambda_2 t}}{\lambda_1-\lambda_2}\right\}. \quad \cdots\cdots (44)$$

If $K^2<\frac{4T_0\gamma g}{T_{10}^2}$, h does damped oscillation, the center of the oscillation is at the height $h_1=\frac{T_{10}\triangle T_0}{\gamma T_0}$.

If $K^2>\frac{4T_0\gamma g}{T_{10}^2}$, the velocity $\frac{dh}{dt}$ increases from o at first as h increases and attains maximum at some height, and afterwards decreases gradually as h increases, and becomes o at the height h_1. h_1 is the half of h_2 given in Table 3, and nearly agrees to the height of cumulus clouds ever observed.

Velocity $\frac{dh}{dt}$ and acceleration $\frac{d^2h}{dt^2}$ are obtained from (44) as the following.

$$\frac{dh}{dt}=\frac{T_{10}\triangle T_0}{\gamma T_0}\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{-\lambda_1 t}-e^{-\lambda_2 t}\right), \quad \cdots\cdots (45)$$

$$\frac{d^2h}{dt^2} = \frac{T_{10}\triangle T_0}{\gamma T_0}\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(\lambda_2 e^{-\lambda_2 t} - \lambda_1 e^{-\lambda_1 t}\right). \quad \cdots\cdots\cdots\cdots \quad (46)$$

Hence we know that $\frac{dh}{dt}$ becomes maximum at the time which satisfies the relation $e^{(\lambda_2-\lambda_1)t} = \frac{\lambda_2}{\lambda_1}$. If we denote that time by t_m, it is given by $t_m = \frac{1}{\lambda_2-\lambda_1} log \frac{\lambda_2}{\lambda_1}$.

To know the rough magnitude of the critical value of K, $K_c = \sqrt{\frac{4T_0\gamma g}{T_{10}^2}}$, we put $\gamma = \frac{1}{200}\frac{^\circ K}{m}$, $T_0 = 300^\circ K$, $T_{10} = 290^\circ K$, and $g = 9.8\frac{m}{sec^2}$, then we obtain

$$K_c = 0.0265\ \frac{1}{sec}. \quad \cdots\cdots\cdots\cdots \quad (47)$$

If there is a spherical mass of air, whose radius and mass are $r\,cm$ and $m\,g$ respectively, and the resistance from the surrounding atmosphere to the ascending sphere is expressed by Stokes law, i. e., equals to $6\pi r\eta\frac{dh}{dt}$, where η is the coefficient of viscosity of the atmosphere, then we have

$$6\pi r\eta = mK \quad \text{and} \quad m = \frac{4}{3}\pi r^3\rho$$

where ρ is the density of air. If the critical value of η, which corresponds to K_c, is expressed by η_c, it is obtained as follows.

$$\eta_c = \frac{2}{9}r^2\rho K_c \fallingdotseq 7.58\times 10^{-6} r^2\ C.G.S. \quad \cdots\cdots\cdots\cdots \quad (48)$$

Three numerical examples of η_c are given in the following.

(i) For the spherical mass of air of $1\,kg$, or the radius $57\,cm$,

$$\eta_c \fallingdotseq 0.0247 C.G.S.$$

(ii) For the spherical mass of radius $100m$,

$$\eta_c = 758\ C.G.S. \fallingdotseq 4\sim 5\times 10^6\,\eta_{mol}.$$

where η_{mol} is the coefficient of molecular viscosity, which is $0.0001710\ C.G.S.$ at $0^\circ C$ and $0.000181\ C.G.S.$ at $20^\circ C$.

(iii) For the spherical mass of air of radius $1\,km$,

$$\eta_c = 75800\ C.G.S. \fallingdotseq 4\sim 5\times 10^8\,\eta_{mol}.$$

η_c thus obtained must be regarded as the coefficient of eddy viscosity and they are nearly 10^6 times as large as the coefficient of molecular viscosity of the air, which are not unreasonable values for atmosphere.

For larger mass of air η_c becomes larger, and so it is supposed that large mass of air would be subjected to the damped oscillation. Practically, a few trains of wave clouds are observed very often when wind blows over mountain range. This means probably that the large mass of air would be subjected to the damped oscillations.

Table 4.

damping ratio	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$K\left(\frac{1}{sec}\right)$	0.0024	0.0057	0.0107	0.0146
$\frac{\eta}{\eta_c}$	0.091	0.215	0.403	0.552

The values of K and η which correspond to some damped oscillations are given in Table 4.

9. Conclusions

The ascension and the changes in state of a small mass of air can be studied by putting proper assumptions. Here we have assumed that the atmosphere is at rest and its temperature changes with height at a constant lapse rate, the ascending mass of air is uniform and has a distinct boundary with the surrounding atmosphere. We have theoretically studied the ascending motion and the changes in state of a small mass of air when the temperature of it is different from that of the surrounding atmosphere.

We have taken up the cases where there is no transference of heat, and there is some transference of heat in proportion to the difference of temperatures between the mass of air and the surrounding atmosphere, the cases where there is no resistance to the motion of the mass of air and there is a resistance proportional to the velocity.

We have been able to obtain the energy equation of the mass of air, the formula which expresses the change in temperature with the ascending motion, the relation between ascending velocity and height, and the limit of ascent of air.

In conclusion, the auther wishes to express his sincere thanks to professor S. T. Nakamura for his important suggestions and his kind guidance and encouragement, and to Mr. A. Onuki for his valuable instructions and discussions.

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(2) See any Book written on the Meteoroloigcal Thermodynamics.

CRITICISM ON THE FUJITA'S OPINIONS ON THE THERMODYNAMICS

Shigeo MATSUMAE

(Received February 21, 1956)

〔I〕 Introduction

Mr. Fujita wrote his treatisis on the Thermodynamics in the Kumamoto Journals of Science but he was mistaken in these papers. Above all, we point out two gross mistakes in the next paragraph.

〔II〕 Criticism on his treatises.

(1) In the case of "Criticism on the expressions of the second law of Thermodynamics."(1)

He insists that the second law of the classical Thermodynamics is expressed by an equal sign in the simple system of two independent state variables which is commonly dealed with in the Thermodynamics.

We quote the passages from his treatise. (p. 31. p. 32. p. 33. Kumamoto Journal of Science. Vol. 1. No. 4. 1954)

"

2. An Expression of the Second Law of Thermodynamics

To acquire the foundation of criticism, the second law of thermodynamics is expressed in the following form.

In the equilibrium system, uniform in pressure p *and temperature* T, *the extensive quantities* S (*entropy*), V (*volume*) *and* $x_1, x_2, \cdots, x_n$ *are given as the independent state variables. The system suffers a small change in which the changes of the independent variables are given by* $dS, dV, dx_1, \cdots, dx_n$, *and absorbs a small heat* δQ *and does a small external work* δW. *Then, by the first and the second laws of thermodynamics, we obtain*

$$\delta Q = dU + \delta W \quad \cdots\cdots (1)$$

$$\delta Q \leqq TdS = dU + pdV + \sum_{i=1}^{n} X_i\, dx_i \quad \cdots\cdots (2)$$

where U *is the internal energy of the system and* X_i *is the i-th extensive variable corresponding to* x_i. *Separating the work* δW *into two parts, the work done by pressure* δW_1 *and the work done by the forces other than pressure* δW_2, *and connecting (1) and (2), we obtain*

$$dU = \delta Q - \delta W_1 - \delta W_2 = TdS - pdV - \sum_{i=1}^{n} X_i\, dx_i \quad \cdots\cdots (3)$$

and

$$dU \leqq TdS - pdV - \delta W_2 \leqq TdS - \delta W_1 - \delta W_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (4)$$

using the relations

$$TdS \geqq \delta Q,\ pdV \geqq \delta W_1,\quad \sum_{i=1}^{n} X_i dx_i \geqq \delta W_2 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (5)$$

Besides, since dU *is a perfect differential, there are relations*

$$T = \left(\frac{\partial U}{\partial S}\right)_{V, x_i},\quad -p = \left(\frac{\partial U}{\partial V}\right)_{S, x_i},\quad -X_i = \left(\frac{\partial U}{\partial x_j}\right)_{S, V, x_j\,(j \neq i)} \quad \cdots\cdots \quad (6)$$

"

For example, we quote the case of "Isothermal changes" here.

"

(iii) Isothermal changes

In this case, if we use the Helmholtz's free energy $F = U - TS$, *we obtain from* (3) *and* (4) *the following expression*

$$dF = -pdV - \sum_{i=1}^{n} X_i dx_i \leqq -pdV - \delta W_2 \leqq -\delta W \quad \cdots\cdots\cdots\cdots \quad (10)$$

The increase in F *is equal to the work done reversibly and is smaller than the work done irreversibly by the outside.*

If $\delta W_2 = 0$, *from* (10) *we obtain an inequality*

$$dF < -pdV \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (11)$$

But this means really

$$dF = -pdV - \sum_{i=1}^{n} X_i dx_i,\quad \sum_{i=1}^{n} X_i dx_i \geqq 0 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (12)$$

and (11) *is looked as the simplified form of* (12). *If, therefore,* $\sum_{i=1}^{n} X_i dx_i = 0$ *is satisfied,* $dF = -pdV$ *holds even when any irreversible volume change is produced. As the inequality of* (11) *is caused from* $\sum_{i=1}^{n} X_i dx_i > 0$, *in the simple system which has only two independent variables,* S *and* V *for example, the relation* $dF = -pdV$ *is always satisfied under the condition* $T = const.$, *even when any irreversible change is produced. There are, however, some textqooks which express that the relation* $dF = -pdV$ *is satisfied only in the reversible volume change, and the relation* $dF < -pdV$ *is satisfied in the irreversible volume change.*

"

(i) It is a mistake that $dF = -pdV$ is applied by Mr. S. Fujita to the irreversible change. The similar mistakes as in this case are found in other cases.

(ii) As $dF = -pdV - \Sigma X_i dx_i$ is obtained from $F = U - TS$ and $dU = TdS - pdV - \Sigma X_i dx_i$ which is given under the condition of the reversible process, $dF = -pdV - \Sigma X_i dx_i$ is applicable only to the reversible change.

In the simple system of two independent state variables, say V and S, we have $\Sigma X_i dx_i = 0$. Consequently, from $\Sigma X_i dx_i = 0$ and $dF = -pdV - \Sigma X_i dx_i$, we get $dF = -pdV$ in the case of the simple system which has two independent state variables. Naturally, $dF = -pdV$ is applicable only to the reversible change.

(2) In the case of "Quasi-stationary process"[2]

We quote a part of his treatise here (p. 51. p. 52. Kumamoto Jourual of Seience Vol. 2. No. 1 (1954))

"

Abstract

It is proved and explained by Prigogine, de Groot, and Meixner that the entropy production of irreversible process is minimum in the stationary state. The writer studied the conditions under which the entropy production of irreversible process is minimum in the stationary state, and could acknowledge that there are some cases where the entropy production is considered not minimum in the stationary state

"

"

In the next paragraph there are stated some simple examples which will be seen as the cases where the entropy productions ate not minimum in the stationary states.

2. Simple Examples of Stationary States in which the Entropy Productions are not Minimum.

a) When an electric circuit is exerted by a fixed E. M. F., some new E. M. F. is induced in the direction opposite to the increase of the electric current, and the current increases from 0 to a final stationary value which is looked upon as rather maximum than minimum.
b) After the electric current in a circuit becomes stationary under a constant E. M. F., if any magnetic perturbation is given in the manner that the induced E. M. F. in the circuit has the direction opposite to the initial current, the entropy production in the perturbed state becomes smaller and shortly afterwards recover to the initial value.

"

(i) Judging from the above sentense, it is certain that Mr. S. Fujita misunderstands the principle of the minimum of the Eutropy-production in the stationary state.
(ii) According to this principle, the electric current in the stationary state takes such a distribution that the total amount of the Joul-heat generated by the electric-current is minimum. For example, when the electric current flows through a passive network, the electric current in the stationary state flows in accordance with the Kirchhoff's law and, in this case, the total amount of the Joul-heat is minimum, if compared with that in other cases. And the Kirchhoff's law is derived reciprocally from the principle of the minimum of the Joul-heat in the stationary state.

"

c) Whern the both ends of a homogeneous straight rod which has uniform section are kept at two constant temperatures T_1 *and* T_2 $(T_1 > T_2)$ *respectively, heat flows from* T_1 *end to* T_2 *end, and if heat do not pass through lateral surface, the temperature* T *of the middle point of the rod is* $\frac{T_1+T_2}{2}$ *in the stationary flow. But it is proved as follows that the entropy production is minimum in the case of* $T=\sqrt{T_1T_2}$ *and not minimum in the case of* $\frac{T_1=T_2}{2}$.

If the sectional area, the length and the heat conductivity are denoted by A, $2l$ *and* k, *the entropy production* σ *caused by the conduction of heat is expressed as following.*

$$\sigma = \frac{kA}{l}\left\{\frac{(T_1-T)^2}{TT_1} + \frac{(T-T_2)^2}{TT_2}\right\} \quad \cdots\cdots (1)$$

Differentiating this with T, *we get*

$$\frac{d\sigma}{dT} = \frac{kA}{l}\left(1-\frac{T_1T_2}{T^2}\right)\left(\frac{1}{T_1}+\frac{1}{T_2}\right) \quad \cdots\cdots (2)$$

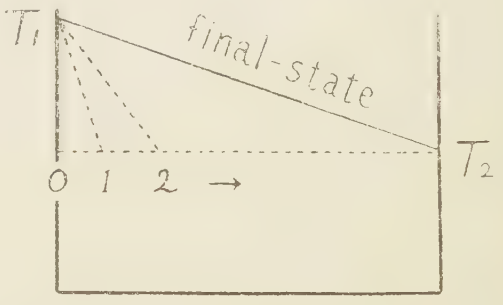
$$\frac{d^2\sigma}{dT^2} = \frac{2kA}{l}\frac{T_1T_2}{T^3}\left(\frac{1}{T_1}+\frac{1}{T_2}\right) \quad \cdots\cdots (3)$$

Thus when T *is* $\sqrt{T_1T_2}$, σ *is minimum, but the flow of heat is not stationary, and therefore, if the initial value of the temperature* T *of the middle point is* $\sqrt{T_1T_2}$, T *rises gradually to the final stationary value* $\frac{T_1+T_2}{2}$, *and* σ *increases. If the initial value of* T *is higher than* $\frac{T_1+T_2}{2}$, T *falls gradually to the final stationary value and* σ *decreases. Thus the value of* σ *in the stationary state is not minimum nor maximum.*

"

(i) He is also mistaken in the above example.

(ii) At first, we assume that one end of the rod is in contact with the heat-source of T_2 ℃ and the rod is hold at the temperature of T_2 ℃. And then, when the other end of the rod is brought in contact with the heat-source of T_1 ℃, the distribution of the temperature in the rod changes gradually from 0-state to 1-state and then from 1-state to 2-state, successively ····· and, at last,



Fig, 1. The change of the temperature-distribution in the rod, in process of time.

comes to be the final state in which the Entropy-production is minimum, if compared with that in other cases.

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(2) Shigeichi-Fujita : "Ouasi-stationary process." Vol. 2. No. 1. Kumamoto Journal of Science (1954)

THE EFFECT OF DEMAGNETIZATION ON THE LONGITUDINAL MAGNETOSTRICTION IN WEAK MAGNETIC FIELD.

Shigeo MATSUMAE

(Received February 11, 1956)

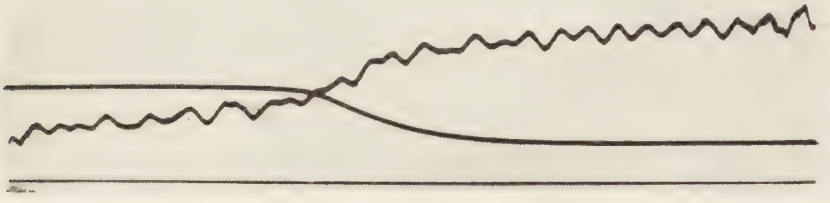
〔I〕 Introduction.

The longitudinal magnetostriction in the weak magnetic field is greatly affected by demagnetizing, as already described,[1][2] We study this phenomenon in some detail.

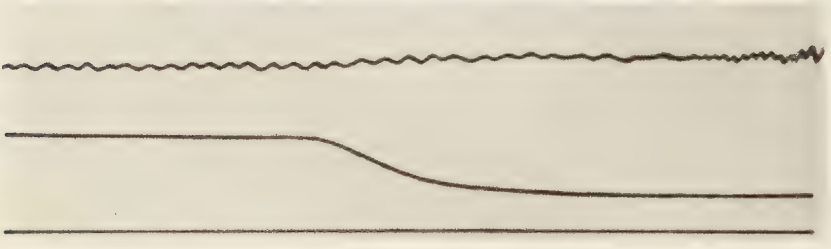
〔II〕 Experimental result.

The experimental process is the same as in the previous papers.[1][2]

The specimens are round bars of pure nickel, 4mm in diameter and 20 cm in length, as in the previous paper.[1] The specimens are cold-worked, set in the electric-furnace, heated from the room-temperature to 800 °C in about 40 minutes, and then hold at this temperature in the course of four different times, 5, 10, 20 and 60 minutes, finally being cooled to the room-temperature. The experimental results on these specimens are described in the previous paper.[1] In all cases, the longitudinal magnetostriction in the weak magnetic field becomes so small as to be almost unobservable, when the specimen is demagnetized with the alternate magnetic field having a small amplitude, say about 7 Oersted.



(a) Before demagnetizing.



(b) After demagnetizing with the alternate magnetic field of the maximum amplitude of about 7 Oersted

Fig. 1. Oscillograph-records of the longitudinal magnetostriction of nickel in weak magnetic field under the compressive stress of 1.2kg/mm^2

The same tendency is also observed, when a compressive stress is applied to the specimen. The specimen is a round bar of nickel, containning a small amount of manganese, 2.5 mm in diameter and 19 cm long, as in the previous paper.[2] The specimen is annealed at 900°C for 90 minutes, cooled to the room-temperature, and then demagnetized with the alternate magnetic field of the maximum amplitude of about 0.7 Oersted.

For example, when the compressive stress of 1.2 kg/mm² is applied to the specimen, the longitudinal magnetostriction in the weak magnetic field is shown in the upper part of the Fig. 1, as already described in the previous paper.[2] But, when the specimen is demagnetized with the alternate magnetic field having a small amplitude, say about 0.7 Oersted, the longitudinal magnetostriction in the weak magnetic field becomes so small as to be slightly observed, as shown in the lower part of Fig. 1. Fig. 2 shows the $\frac{dl}{l} - I$ curves in these two cases.

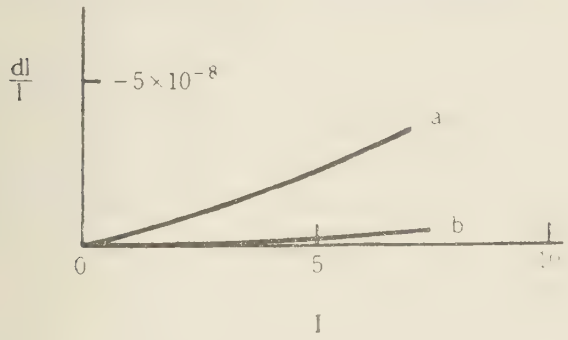


Fig. 2. $I - \frac{dl}{l}$ curves of the above oscillograph-records.
(a). Before demagnetizing.
(b). After demagnetizing.

〔III〕 Conclusion.

In nickel, the longitudinal magnetostriction in the weak magnetic field decreases greatly by demagnetizing the specimen even with the alternate magnetic field of a small amplitude, when it exists.

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ON THE ADIABATIC CHANGE OF COMPLETELY IONIZED GASES IN THE STARS

Keisuke KAMINISI

(Received February 28, 1956)

Abstract

Taking account of electrostatic interaction among particles and radiation, we have calculated the effective polytropic indices n_{eff} in the adiabatic changes of gases. The present results show that both electrostatic interaction and radiation result to *increase* the value of n_{eff}.

1. Introduction

At present it is considered that convective equilibrium plays an important role in the interior of main-sequence stars. Since in the convective zone the material is approximately considered to move adiabatically, the most fundamental quantity in the study of structure of convective region is adiabatic exponent Γ_2, govering adiabatic changes. As for the value of Γ_2 (or of the effective polytropic index n_{eff}) many astrophysicists have adopted the relation $\Gamma_2 = \frac{5}{3}$ (or $n_{eff} = \frac{1}{\Gamma_2 - 1} = 1.5$), which is valid for an enclosure containing a perfect monoatomic gas and no radiation. However in the stellar interior atoms are highly ionized and there exist Coulomb interactions among the particles. Considering such a condition, R. E. Williamson discussed the problem of adiabatic exponent of ionized gas [1], but he restricted himself to the only one case of hydrogen gas and did not take account of the energy of radiation. A more general extension is presented in this paper.

2. Pressure and Internal Energy of Completely Ionized Gases

It is well known that for all regions in which we may apply the classical statistical mechanics the gas pressure p_g is, by the virial theorem, expressed as follows:

$$p_g V = NkT + \frac{1}{3} \text{ (internal virial)} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (1),$$

where N is the number of particles contained in the volume V, T is the absolute temperature, and k is Boltzmann's constant.

Now we consider the ionized gas in a star. Here the forces between the particles are the electrostatic attractions and repulsions of their charges. Since these are inverse-square force, the internal virial is equal to the potential energy v in the volume V.

The problem of determining v when it is sufficiently small compared with $p_g V$ has

been treated by S. Rosseland [2] following the theory developed by Debye and Hückel for ionized solutions. When we consider a completely ionized gas consisting of a single chemical element of atomic number Z, Coulomb potential energy v is, according to Rosseland's calculation [2], given by

$$v = -\left(\frac{\pi Z^3 (Z+1)^3 e^6 n_i^2}{kTV}\right)^{1/2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (2),$$

where e is a unit charge in e. s. u., and n_i is the number of ions contained in the volume V. Although the relation (2) has been derived by neglecting the square of $Ze\varphi / kT$ (where φ is the interaction potential between two particles), such an approximation is fairly accurate for all regions in which atoms are completely ionized.

Now if we consider a gas consisting of completely ionized atoms of atomic number Z and of atomic weight A, we can approximately get the following relations:

$$N = \frac{m}{AH}(1+Z) \text{ and } n_i = \frac{m}{AH} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3),$$

where m is the mass in volume V and H is the mass of hydrogen atom. From the equations (1), (2) and (3), we can finally obtain the following equation of state:

$$P = p_i + p_r = \frac{m}{AH}\frac{(1+Z)kT}{V} - \frac{1}{3}\left(\frac{\pi Z^3 (Z+1)^3 e^6 m^3}{A^3 H^3 k}\right)^{1/2} V^{-3/2} T^{-1/2} + \frac{1}{3} aT^4 \quad (4),$$

where p_r is the radiation pressure and a is the radiation density constant.

Next from the eq. (4) we can find the internal energy U in our system, by integrating the following well-known formula of thermodynamics:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (5),$$

that is

$$U = \frac{3}{2}\frac{m}{AH}(1+Z)kT - \left(\frac{\pi Z^3 (Z+1)^3 e^6 m^3}{A^3 H^3 k}\right)^{1/2} V^{-1/2} T^{-1/2} + aVT^4 \quad \cdots\cdots \quad (6),$$

where the integral constant which is a function of T has been determined by considering the value of U as density $\rho = \frac{m}{V} \to 0$.

3. Expression for the Effective Polytropic Index

The effective polytropic index n_{eff} is defined by the following relation [3]:*

* It should be noted that n_{eff} defined by (7) is, in the case of perfect gas, equivalent to n_{eff} by the following usual definition [4]:

$$\frac{dP}{P} = \left(1 + \frac{1}{n_{eff}}\right)\frac{d\rho}{\rho} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (7'),$$

however in the case of imperfect gas the definition (7) will be slightly different from the definition (7').

$$n_{eff}+1=\frac{\Gamma_2}{\Gamma_2-1}=\frac{d\log P}{d\log T} \quad \cdots\cdots (7).$$

Now in our system we regard T and V as the independent state variables, then we have

$$dP=\left(\frac{\partial P}{\partial V}\right)_T dV+\left(\frac{\partial P}{\partial T}\right)_V dT \quad \cdots\cdots (8).$$

On the other hand the first law of thermodynamics shows that for an adiabatic change,

$$0=dQ=\left(\frac{\partial U}{\partial T}\right)_V dT+\left\{\left(\frac{\partial U}{\partial V}\right)_T+P\right\}dV \quad \cdots\cdots (9).$$

By making use of eqs. (5), (8) and (9), we get the following relation:

$$\frac{dP}{P}=\frac{-\left(\frac{\partial P}{\partial V}\right)_T\left(\frac{\partial U}{\partial T}\right)_V+T\left(\frac{\partial P}{\partial T}\right)_V}{P\left(\frac{\partial P}{\partial T}\right)_V}\frac{dT}{T} \quad \cdots\cdots (10).$$

By comparing eq. (7) with (10), we can finally obtain the following expression for the effective polytropic index:

$$n_{eff}+1=\frac{T\left(\frac{\partial P}{\partial T}\right)_V-\left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial P}{\partial V}\right)_T}{P\left(\frac{\partial P}{\partial T}\right)_V} \quad \cdots\cdots (11).$$

Hence from the above relation, we can derive the expression for n_{eff} as a function of T and V (or density ρ) by making use of eqs. (4) and (6). The result is as follows:

$$n_{eff}+1=\frac{16-12\beta-\frac{3}{2}\beta^2+F\left(-\frac{32}{3}+10\beta+\frac{3}{4}\beta^2\right)+F^2\left(\frac{16}{9}-2\beta\right)}{4-3\beta+F\left(-\frac{8}{3}+\frac{5}{2}\beta\right)+F^2\left(\frac{4}{9}-\frac{1}{2}\beta\right)} \quad (12),$$

where $\beta=\frac{p_g}{P}$ which has been introduced for the convenience of astrophysical consideration, and

$$F=\left(\frac{\pi Z(Z+1)e^2m}{AHk^2T}\right)^{\frac{1}{2}}=\left(\frac{\pi Z(Z+1)e^2\rho}{AHk^3T^3}\right)^{\frac{1}{2}} \quad \cdots\cdots (13)$$

which is the ratio of the internal virial to the kinetic energy of the particles in our system.

If F tends to zero, the expression (12) turns to the following formula:

$$n_{eff}+1=\frac{32-24\beta-3\beta^2}{2(4-3\beta)} \quad \cdots\cdots (14),$$

which is just the expresson in the case of perfect gas [5].

In usual stars, the ratio of the radiation pressure to the total pressure is very small,

that is $1-\beta \ll 1$, and F is also considerably small. Hence it is useful for us to simplify the expression (12) by neglecting the second order of both $(1-\beta)$ and F. Then we can get the following approximate formula:

$$n_{eff} = \frac{3}{2} + \frac{15}{2}(1-\beta) + \frac{1}{2}F \quad \cdots\cdots (15).$$

In the above relation we find that both radiation energy and Coulomb interaction energy result to increase the value of n_{eff}.

4. Numerical Value of the Effective Polytropic Index

Since it is considered that the gases in usual stars mainly consist of hydrogen and helium, we shall now consider the hydrogen gas and helium gas in this section. Then the values of F are as follows:

$F_H = 0.1322\left\{\rho/(T/10^6)^3\right\}^{1/2}$ in the case of hydrogen gas, and $F_{He} = 0.2290\left\{\rho/(T/10^6)^3\right\}^{1/2}$ in the case of helium gas.

Using these values, we can find the correction values (which are to be added to 1.5) caused by Coulomb interaction. These values are tabulated in the table for various values of

TABLE

The Correction Values for the Effective Polytropic Index

(We get the value of n_{eff} by adding each value of this table to 1.5.)

ρ \ $T/10^6$		1	3	5	10	20	40	60
1	Coulomb H	0.066	0.013	0.005 9	0.002 1	0.000 74	0.000 26	0.000 14
	He	0.12	0.022	0.010	0.003 6	0.001 3	0.000 45	0.000 25
	Radiation H	0.000 12	0.003 1	0.014	0.11	0.60	1.24	1.47
	He	0.000 31	0.008 2	0.038	0.26	1.0	1.45	1.50
5	Coulomb H	×	0.028	0.013	0.004 7	0.001 7	0.000 55	0.000 32
	He		0.049	0.023	0.008 1	0.002 9	0.001 0	0.000 55
	Radiation H	0.000 023	0.000 62	0.000 29	0.023	0.17	0.80	1.25
	He	0.000 061	0.001 7	0.007 6	0.060	0.38	1.2	1.43
10	Coulomb H	×	0.040	0.019	0.006 6	0.002 3	0.000 78	0.000 45
	He		0.070	0.032	0.011 5	0.004 1	0.001 4	0.000 78
	Radiation H	0.000 012	0.000 31	0.001 4	0.011	0.087	0.52	1.01
	He	0.000 031	0.000 82	0.003 8	0.030	0.21	0.92	1.32
30	Coulomb H	×	×	0.032	0.011	0.004 1	0.001 4	0.000 78
	He			0.056	0.020	0.007 0	0.002 5	0.001 4
	Radiation H	0.000 003 8	0.000 10	0.000 48	0.003 8	0.030	0.22	0.56
	He	0.000 010	0.000 28	0.001 3	0.010	0.078	0.48	0.97
60	Coulomb H	×	×	×	0.016	0.005 7	0.001 9	0.001 1
	He				0.028	0.009 9	0.003 5	0.001 9
	Radiation H	0.000 001 9	0.000 052	0.000 24	0.001 9	0.015	0.11	0.34
	He	0.000 005 1	0.000 14	0.000 64	0.005 1	0.040	0.28	0.68
100	Coulomb H	×	×	×	0.021	0.007 4	0.002 6	0.001 4
	He				0.036	0.013	0.004 5	0.002 5
	Radiation H	0.000 001 14	0.000 031	0.000 14	0.001 1	0.009 2	0.070	0.22
	He	0.000 003 1	0.000 082	0.000 38	0.003 1	0.024	0.18	0.48

The empty section denoted by the symbol × is correspond to the region in which an assumption of the completely ionized gas is not permissible.

ρ and T. As for the correction values caused by radiation we have calculated them by using relation (15) in the case of the lower temperature $(1-\beta \ll 1)$ and relation (12) in the case of the higher temperature $(\beta \ll 1, F \ll 1)$. These values are also tabulated in our table.

5. Conclusion

Under the usual conditions of the stars, the deviations of values of n_{eff} from 1.5 come to several per cent as we see in our table. Hence we must take account of these effect for the more precise investigation of the structure of convective region.

Now we shall point out some features of the present results:

(i) The denser and the cooler the material is, the bigger the correction value caused by Coulomb interaction and the smaller the correction by radiation.

(ii) The heavier the chemical element contained in the system is, the bigger the correction by Coulomb interaction (by the factor $\left[\frac{Z^3(Z+1)}{A}\right]^{1/2}$) and also the bigger the correction by radiation (by the factor $\frac{A}{1+Z}$ in the case of $(1-\beta) \ll 1$, and in the more complicated manner in the case of $\beta \ll 1$).

The auther wishes to express his thanks to the professors of the Department of Physics, Kumamoto University, for their constant encouragement.

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A PROBLEM ON THE SEISMIC PROSPECTING

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(Received February 28, 1956)

1. Introduction.

The most fundamental principle on the refraction method concerning the seismic prospecting which is used at present is a fact that if the curve of separation is a straight line, the travelling-time curve is also a straight line, and making the travelling-time curve by experiment, we can calculate the curve of separation and the velocity of wave in the inner layer. (1)

But when the travelling-time curve is not a straight line, how may we explore? At this point of view, in the previous Journal, I introduced general fundamental relations between the curve of separation and the travelling-time curve. In this paper, by using these relations, I discussed one method to find the form of the separation curve and the velocity of wave in the inner layer when the travelling-time curve is not straight provided that the travelling-time curve and the velocity of wave in the first layer are known.

2. Determination of the Form of Separation Curve.

As shown in Fig. 1, let

$$y=f(x)=y(x)\cdots\quad(1)$$

be the equation of the curve of separation, where y-axis is taken towards under-ground, and

$$t=\varphi(X)\cdots\cdots\quad(2)$$

be the travelling-time curve. Let $OPQR$ be the path along which a wave ray is propagated from O to R, from O to P with the velocity v_1 and at P the incident angle is equal to the critical angle θ and then the wave runs along the curve $y=f(x)$ from P to Q with the velocity v_2 and at Q the wave refracts at

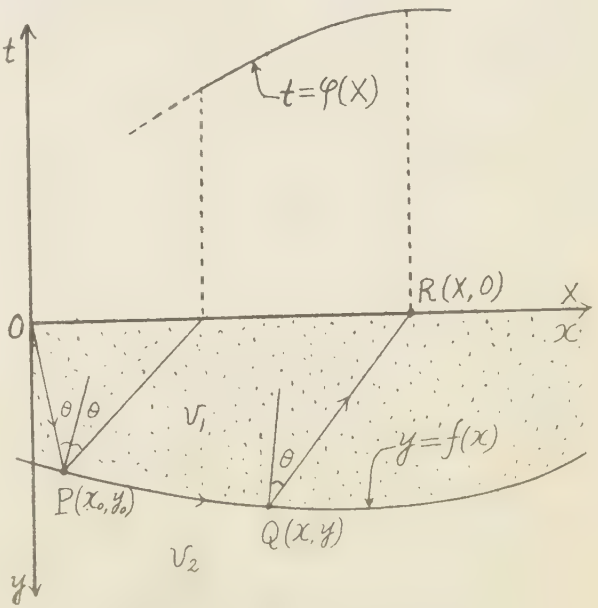

Fig. 1.

the same angle θ and runs from Q to R with the velocity v_1, then we have following relations which were shown in the preceding Journal

$$t = \frac{v_2\sqrt{1+f'(x_0)^2}}{v_1[v_1-Vf'(x_0)]}x_0 + \frac{1}{v_2}\int_{x_0}^{x}\sqrt{1+f'(x)^2}\,dx + \frac{v_2\sqrt{1+f'(x)^2}}{v_1[V-v_1f'(x)]}f(x) = \varphi(X) \quad (3)$$

$$X = x + \frac{v_1+Vf'(x)}{V-v_1f'(x)}f(x) \quad \cdots\cdots (4)$$

$$v_1 + Vf'(x) - v_1v_2\sqrt{1+f'(x)^2}\,\varphi'(X) = 0 \quad \cdots\cdots (5)$$

$$x_0 = \frac{v_1-Vf'(x_0)}{V+v_1f'(x_0)}f(x_0) \quad \cdots\cdots (6),$$

where v_1 and v_2 are the velocities of wave in the first layer and the second layer respectively, and $v_2 > v_1$, $V = \sqrt{v_2^2-v_1^2}$.

Now, let us suppose that $|f''(x)|$ is very small, that is the curve of separation is near to a straight line, then the travelling-time curve will be near to a straight line also, and suppose that the function $\varphi(X)$ is approximated by a quadratic formula as

$$t = \varphi(X) = \frac{1}{2}aX^2 + sX + c'' \quad \cdots\cdots (7),$$

and let the straight line drawn approximately to (7) be

$$\varphi_0(X) = bX + c' \quad \cdots\cdots (8).$$

Next, let the solutions of (7) and (8) be

$$y(x) = \tau(x) + \beta x + c \quad \cdots\cdots (9)$$

$$y_0(x) = \beta x + c_0 \quad \cdots\cdots (10)$$

respectively, then we can assume to

$$\left.\begin{aligned} &a = \mathrm{O}(h), \qquad s-b = \mathrm{O}(h), \quad c''-c' = \mathrm{O}(h) \\ &\tau(x) = \mathrm{O}(h), \quad \tau'(x) = \mathrm{O}(h) \end{aligned}\right\} \quad \cdots\cdots (11)$$

where h is a small value.

At first, as (10) is the solution of (8), from (5) we have

$$\alpha + \beta = \alpha v_2 b\gamma \quad \cdots\cdots (12)$$

where $\alpha = v_1/V$, $\gamma = \sqrt{1+\beta^2}$, therefore

$$\alpha v_2 s = \alpha v_2 b + \mathrm{O}(h) = \frac{\alpha+\beta}{\gamma} + \mathrm{O}(h) \quad \cdots\cdots (13).$$

Next, neglecting $O(h^2)$, we get

$$\left.\begin{aligned} &y'(x)^2 = \beta^2 + 2\beta\tau'(x) = 2\beta y'(x) - \beta^2, \ \sqrt{1+y'(x)^2} = \frac{1}{\gamma}[\gamma^2 + \beta\tau'(x)] \\ &\varphi'(X) = aX + s = \frac{a}{1-\alpha\beta}[\gamma^2 x + (\alpha+\beta)c] + s, \end{aligned}\right\} \quad \cdots\cdots (14)$$

substituting these relations in (5), we get

$$k(\gamma-\alpha\beta v_2 s)\frac{dy}{dx}=a\alpha v_2\gamma^4 x+a\alpha v_2\gamma^2(\alpha+\beta)c+\alpha v_2 sk-\alpha\gamma k \quad \cdots\cdots\cdots\cdots \quad (15)$$

where $k=1-\alpha\beta$,
therefore putting to

$$A'=k(\gamma-\alpha\beta v_2 s)$$

$$F'=\frac{1}{2}a\alpha v_2\gamma^4$$

$$K'=a\alpha v_2\gamma^2(\alpha+\beta)c+\alpha v_2 sk-\alpha\gamma k$$

gives

$$A'y(x)=F'x^2+K'x+c'''$$

While

$$A'=k\left[\gamma-\frac{\alpha+\beta}{\gamma}\beta+0(h)\right]=\frac{k^2}{\gamma}+0(h),$$

therefore if $A'=0$, k must be zero and $\beta=1/\alpha=V/v_1$, and applying to (12), we get $v_1 b=1$ which means that (10) has no solution[2].

Accordingly A' is not equal to zero, and putting to

$$F'/A'=F, \qquad K'/A'=K$$

gives

$$F=\frac{\frac{1}{2}a\alpha v_2\gamma^4}{\frac{k^2}{\gamma}+0(h)}=\frac{a}{2k^2}\alpha v_2\gamma^5=0(h)$$

$$K-\beta=\frac{K'}{A'}-\beta=\frac{1}{k^2}\alpha v_2\gamma^3[a(\alpha+\beta)c+k(s-b)]=0(h).$$

Therefore putting to

$$\left.\begin{aligned}F&=\frac{a}{2k^2}\alpha v_2\gamma^5\\ E&=\frac{1}{k^2}\alpha v_2\gamma^3[a(\alpha+\beta)c+k(s-b)]\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (16)$$

we have

$$y(x)=Fx^2+(E+\beta)x+c \quad \cdots\cdots\cdots\cdots \quad (17)$$

and this is our required solution.

Of course the values of v_2 and c are not yet known provided that a, s, c'', c', v_1, b are known and we must determine them.

3. Determination of v_2 and c.

When

$$\left.\begin{aligned} y(x) &= \tau(x) + \beta x + c \qquad e = \mathrm{O}(h) \\ \tau(x) &= Fx^2 + Ex + e, \quad F = \mathrm{O}(h), \quad E = \mathrm{O}(h) \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (18)$$

are given, we get

$$t_1 = \frac{v_2\sqrt{1+y'(x_0)^2}}{v_1[v_1 - Vy'(x_0)]}x_0 = \frac{v_2 x_0}{\gamma v_1 T^2}[\gamma^2 T + (V+\beta v_1)(2Fx_0 + E)]$$

$$t_2 = \frac{1}{v_2}\int_{x_0}^{x}\sqrt{1+y'(x)^2}\,dx = \frac{1}{\gamma v_2}[\beta F x^2 + (\gamma^2+\beta E)x - \{\beta F x_0^2 + (\gamma^2+\beta E)x_0]$$

$$t_3 = \frac{v_2\sqrt{1+y'(x)^2}}{v_1[V - v_1 y'(x)]}y(x) = \frac{v_2}{\gamma v_1 S^2}[\{(1+3\beta^2)V + (1-\beta^2)\beta v_1\}Fx^2$$

$$+ \{\gamma^2\beta(V-\beta v_1) + 2(V\beta + v_1)cF + (\overline{1+2\beta^2}\,V - \beta^3 v_1)E\}x + \{\gamma^2(V-\beta v_1) + (V\beta+v_1)E\}c]$$

where $S = V - \beta v_1$, $T = v_1 - \beta V$. Next, applying (18) in (6) gives

$$(v_1 + 3\beta V)Fx_0^2 + (\gamma^2 + 2cF + 2\beta E)Vx_0 + (V\beta - v_1 + VE)c = 0$$

and solving this equation with respect to x_0, we get

$$x_0 = \bar{x}_0 + \bar{\varepsilon} \quad \cdots\cdots\cdots\cdots \quad (19)$$

where

$$\left.\begin{aligned} \bar{x}_0 &= \frac{v_1 - V\beta}{\gamma^2 V}c \\ \bar{\varepsilon} &= \frac{c}{\gamma^4 V}[(\beta^2-1)V - 2\beta v_1]E + \frac{c^2}{\gamma^6 V^3}[(2-\beta^2)\beta V^3 + (3\beta^2-2)v_1 V^2 - \beta v_1^2 V - v_1^3]F \end{aligned}\right\} \quad (20).$$

Therefore

$$t_1 = \frac{v_2 c}{\gamma^2 v_1 V} - \frac{v_2 c}{\gamma^2 v_1}\left[\frac{c}{\gamma^4 V^3}(v_1^2 - \beta^2 V^2)F + \frac{\beta}{\gamma^2 V}E\right]$$

$$\beta F x_0^2 + (\gamma^2+\beta E)x_0 = \frac{T}{V}c - \frac{V+\beta v_1}{\gamma^2 V}cE + \frac{c^2 F}{\gamma^4 V^3}[2\beta V^3 + (\beta^2-2)v_1 V^2 - v_1^3]$$

and

$$\begin{aligned} t &= \varphi(X) = t_1 + t_2 + t_3 \\ &= AFx^2 + (B + DE + cFG)x + Hc + JEc + LFc^2 \end{aligned} \quad \cdots\cdots\cdots \quad (21)$$

where

$$\left.\begin{aligned} A &= \frac{1}{\gamma\alpha v_2 k^2}(1+3\beta^2+2\alpha\beta-\alpha\beta^3+\gamma^2\alpha^2+\alpha^3\beta), \quad B = \frac{1}{\alpha v_2 k}(\alpha+\beta) \\ D &= \frac{1}{\gamma\alpha v_2 k^2}(1+2\beta^2+\alpha\beta-\alpha\beta^3+\alpha^2), \qquad G = \frac{2}{\gamma\alpha v_2 k^2}(\alpha+\beta)(1+\alpha^2) \end{aligned}\right\} \quad (22)$$

$$H=\frac{1}{\gamma\alpha v_2 k}(\alpha^2+\beta^2+2), \qquad L=\frac{1}{\gamma\alpha v_2}(\alpha-\beta)$$
$$J=\frac{1}{\gamma^3\alpha v_2 k^2}(2\alpha+\alpha^2+\beta^2-\alpha^2\beta+3\alpha\beta^2+2\alpha^3\beta)$$

From (4), we get

$$(\alpha+3\beta+\alpha^2\beta-\alpha\beta^2)Fx^2+[k\gamma^2+(\alpha+2\beta-\alpha\beta^2)E+2(1+\alpha^2)cF]x$$
$$+k(\alpha+\beta)c+(1+\alpha^2)cE-k^2X=0 \quad \cdots\cdots \quad (23)$$

and solving this equation with respect to x gives

$$x=\frac{1}{\gamma^2}[kX-(\alpha+\beta)c]+\delta \quad \cdots\cdots \quad (24)$$

where

$$\left.\begin{aligned}
&\delta=MFX^2+(NFc+PE)X+WFc^2+QEc\\
&M=\frac{-1}{\gamma}k(\alpha+3\beta+\alpha^2\beta-\alpha\beta^2), \quad N=\frac{-2}{\gamma^2}(1+4\alpha\beta-2\beta^2+\alpha^2\beta^2+\alpha\beta^3-\alpha^3\beta)\\
&W=\frac{1}{\gamma^6}(\alpha+\beta)(2-2\alpha\beta+\alpha^2-\beta^2), \quad P=\frac{-1}{\gamma^4}(\alpha+2\beta-\alpha\beta^2)\\
&Q=\frac{-1}{\gamma^4}(1-2\alpha\beta-\beta^2)
\end{aligned}\right\} \quad (25).$$

Substituting these relations in (21), we get

$$t=\varphi(X)=\frac{k^2}{\gamma^5\alpha v_2}FX^2+\left[\frac{\alpha+\beta}{\gamma\alpha v_2}+\frac{k}{\gamma^3\alpha v_2}E-\frac{2(\alpha+\beta)}{\gamma^5\alpha v_2}kFc\right]X$$
$$+\frac{2c}{\gamma\alpha v}-\frac{2\beta}{\gamma^3\alpha v}cE+\frac{2(\alpha^2+\beta^2)}{\gamma^5\alpha v}c^2F \quad \cdots\cdots \quad (26).$$

Here if we put to

$$F=\frac{1}{2k^2}a\alpha v_2\gamma^5, \quad E=\frac{1}{k^2}\alpha v_2\gamma^3[a(\alpha+\beta)c+k(s-b)]$$

(26) becomes

$$t=\varphi(X)=\frac{a}{2}X^2+sX+\frac{2\gamma c}{\alpha v_2 k}-\frac{2\beta sc}{k}+\frac{ac^2}{k^2}(\alpha^2-2\alpha\beta-\beta^2) \quad \cdots\cdots \quad (27)$$

The equations (7) and (27) must express same curve and we get

$$\frac{2\gamma c}{\alpha v_2 k}-\frac{2\beta sc}{k}+\frac{ac^2}{k^2}(\alpha^2-\beta^2-2\alpha\beta)=c'' \quad \cdots\cdots \quad (i)$$

Next, as shown in Fig. 2, let us experiment from opposite direction taking the origin at $(l, 0)$ and let

$$t=\bar{\varphi}(\bar{X})=\frac{\bar{a}}{2}\bar{X}^2+s\bar{X}+c \quad \cdots\cdots \quad (28)$$

be the travelling-time curve and

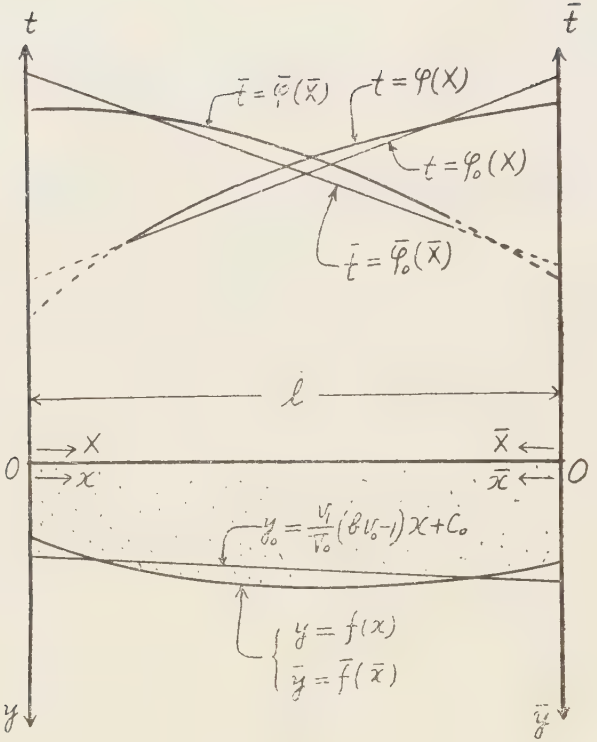

Fig. 2.

$$\bar{\varphi}_0(\bar{X}) = \bar{b}\bar{X} + \bar{c}' \quad \cdots\cdots (29)$$

be the straight line drawn approximately to (28), then, similarly as above, the solution of (28) becomes

$$y(x) = \bar{F}x^2 + (\bar{E}-\beta)\,\bar{x} + \bar{c} \quad \cdots\cdots (30)$$

where

$$\left.\begin{aligned} \bar{F} &= \frac{1}{2\bar{k}^2}\,\bar{a}\alpha v_2\gamma^5 \\ \bar{E} &= -\frac{1}{\bar{k}^2}\,\alpha v_2\gamma^3[\bar{a}(\alpha-\beta)\bar{c} + \bar{k}(\bar{s}-\bar{b})] \\ \bar{k} &= 1 + \alpha\beta \end{aligned}\right\} \quad \cdots\cdots (31).$$

The expressions (17) and (30) must express same curve and we have $y(l) = \bar{y}(0)$, that is

$$Fl^2 + (E+\beta)l + c = \bar{c} \quad \cdots\cdots (ii).$$

Also as corresponding relation to (i), we get

$$\frac{2\gamma\bar{c}}{\alpha v_2\bar{k}} + \frac{2\beta\bar{s}\bar{c}}{\bar{k}} + \frac{\bar{a}\bar{c}^2}{\bar{k}^2}(\alpha^2-\beta^2+2\alpha\beta) = \bar{c}'' \quad \cdots\cdots\cdots\cdots \text{(iii)}.$$

Consequently we have following equations

$$\left.\begin{aligned}
&\alpha v_2 b\gamma - (\alpha+\beta) = 0\\
&\frac{a}{2}\alpha v_2\gamma^5 l^2 + \alpha v_2\gamma^3[a(\alpha+\beta)c + (1-\alpha\beta)(s-b)]l\\
&\qquad + (1-\alpha\beta)^2(\beta l + c - \bar{c}) = 0\\
&\frac{2\gamma c}{\alpha v_2(1-\alpha\beta)} - \frac{2\beta sc}{1-\alpha\beta} + \frac{ac^2}{(1-\alpha\beta)^2}(\alpha^2-\beta^2-2\alpha\beta) - c'' = 0\\
&\frac{2\gamma\bar{c}}{\alpha v_2(1+\alpha\beta)} + \frac{2\beta\bar{s}\bar{c}}{1+\alpha\beta} + \frac{\bar{a}\bar{c}^2}{(1+\alpha\beta)^2}(\alpha^2-\beta^2+2\alpha\beta) - \bar{c}'' = 0
\end{aligned}\right\} \quad \cdots\cdots\cdots \text{(32)}$$

and we can obtain the values of $\alpha, \beta, c, \bar{c}$ by solving these equations simultaneously provided that a, s, c'', $\bar{a}$, $\bar{s}$, $\bar{c}''$, b, l, v_1 are known. While $\gamma = \sqrt{1+\beta^2}$, $\alpha v_2 = v_1 v_2/V = v_1\sqrt{1+\alpha^2}$, therefore we may take α, β, c, $\bar{c}$ as unknowns.

Generally, it is very difficult to solve (32) analytically, but to find the numerical approximate solutions of them is not so difficult. Let $\alpha_0, \beta_0, c_0, \bar{c}_0$ be a set of approximate values of α, β, c, $\bar{c}$ respectively and if we put to

$$\alpha = \alpha_0 + \delta, \quad \beta = \beta_0 + \varepsilon, \quad c = c_0 + \sigma, \quad \bar{c} = \bar{c}_0 + \bar{\sigma} \quad \cdots\cdots\cdots\cdots \text{(33)}$$

δ, ε, σ, $\bar{\sigma}$ are very small. Therefore substituting (33) in (32) and neglecting high order of δ, ε, σ, $\bar{\sigma}$ we get

$$\left.\begin{aligned}
&G_1\delta + H_1\varepsilon \cdots\cdots\cdots + N_1 = 0\\
&G_2\delta + H_2\varepsilon + L_2\sigma + M_2\bar{\sigma} + N_2 = 0\\
&G_3\delta + H_3\varepsilon + L_3\sigma \cdots\cdots + N_3 = 0\\
&G_4\delta + H_4\varepsilon \cdots\cdots + M_4\bar{\sigma} + N_4 = 0
\end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \text{(34)}$$

where

$$\gamma_0 = \sqrt{1+\beta_0^2}, \quad \kappa_0 = \sqrt{1+\alpha_0^2}$$

$$G_1 = \frac{1}{\kappa_0} v_1 b\alpha\gamma - 1, \quad H_1 = \frac{1}{\gamma_0} v_1 b\beta\kappa - 1$$

$$N_1 = v_1 b\kappa_0\gamma_0 - (\alpha_0+\beta_0)$$

$$G_2 = \frac{v_1\gamma_0^3 l}{\kappa_0}\left[\frac{a}{2}\alpha_0 b\gamma_0^2 + ac_0(1+2\alpha_0^2+\alpha_0\beta_0) + (s-b)(\alpha_0-\beta_0-2\alpha_0^2\beta_0)\right]$$
$$-2(1-\alpha_0\beta_0)\beta_0(l\beta_0+c_0-\bar{c}_0)$$

$$H_2 = v_1 l\kappa_0\gamma_0\left[\frac{5}{2}al\gamma_0^2\beta_0 + ac_0(1+3\alpha_0+4\beta_0^2) + (s-b)(3\beta_0-\alpha_0-4\alpha_0\beta_0^2)\right]$$
$$+(1-\alpha_0\beta_0)[l(1-3\alpha_0\beta_0)-2\alpha_0(c_0-\bar{c}_0)]$$

$$
\left.\begin{aligned}
L_2 &= av_1 l\kappa_0\gamma_0^2(\alpha_0+\beta_0)+(1-\alpha_0\beta_0)^2,\quad M_2=-(1-\alpha_0\beta_0)^2\\
N_2 &= v_1 l\gamma_0^2\kappa_0\left[\frac{a}{2}\,l\gamma_0^2+ac_0(\alpha_0+\beta_0)+(s-b)(1-\alpha_0\beta_0)\right]\\
&\qquad +(1-\alpha_0\beta_0)^2(l\beta_0+c_0-\bar{c}_0)\\
G_3 &= 2s\beta_0^2c_0-\frac{2\gamma_0c_0}{v_1\kappa_0^2}(\alpha_0+\beta_0)+2ac_0^2(\alpha_0-\beta_0)+2c''(1-\alpha_0\beta_0)\beta_0\\
H_3 &= \frac{2c_0}{v_1\kappa_0\gamma_0}(\beta_0-\alpha_0-2\alpha_0\beta_0^2)+2sc_0(2\alpha_0\beta_0-1)-2ac_0^2(\alpha_0+\beta_0)+2c''(1-\alpha_0\beta_0)\alpha_0\\
L_3 &= (1-\alpha_0\beta_0)\left(\frac{2\gamma_0}{v_1\kappa_0}-2s\beta_0\right)+2ac_0(\alpha_0^2-\beta_0^2-2\alpha_0\beta_0)\\
N_3 &= (1-\alpha_0\beta_0)\left(\frac{2\gamma_0}{v_1\kappa_0}-2s\beta_0\right)c_0+ac_0^2(\alpha_0^2-\beta_0^2-2\alpha_0\beta_0)-c''(1-\alpha_0\beta_0)^2\\
G_4 &= 2\bar{s}\bar{\beta}_0^2\bar{c}_0+\frac{2\gamma_0\bar{c}_0}{v_1\kappa_0^2}(\beta_0-\alpha_0)+2\bar{a}\bar{c}_0(\alpha_0+\beta_0)-2\bar{c}''(1+\alpha_0\beta_0)\beta_0\\
H_4 &= \frac{2\bar{c}_0}{v_1\kappa_0\gamma_0}(\alpha_0+\beta_0+2\alpha_0\beta_0^2)+2\bar{s}\bar{c}_0(1+2\alpha_0\beta_0)+2\bar{a}_0\bar{c}_0^2(\alpha_0-\beta_0)-2c''(1+\alpha_0\beta_0)\alpha_0\\
M_4 &= (1+\alpha_0\beta_0)\left(\frac{2\gamma_0}{v_1\kappa_0}+2\bar{s}\beta_0\right)+2\bar{a}\bar{c}_0(\alpha_0^2-\beta_0^2+2\alpha_0\beta_0)\\
N_4 &= (1+\alpha_0\beta_0)\left(\frac{2\gamma_0}{v_1\kappa_0}+2\bar{s}\beta_0\right)\bar{c}_0+\bar{a}\bar{c}_0^2(\alpha_0^2-\beta_0^2+2\alpha_0\beta_0)-\bar{c}''(1+\alpha_0\beta_0)^2
\end{aligned}\right\}\quad(35)
$$

and solving (34) simultaneously with respect to δ, ϵ, σ, $\bar{\sigma}$ gives the values of v_2, β, c, $\bar{c}$.

A method to find a set of α_0, β_0, c_0, $\bar{c}_0$ is as follows. Take (8) and (29) as a set of travelling-time curves and solve them by ordinary method provided that v_1 is known and let the solutions of $y(x)$ and v_2 be $y_0(x)=\beta_0x+c_0$, v_0, then $\alpha_0=\sqrt{v_0^2-v_1^2}$, β_0, c_0 are a set of approximate values of α, β, c and $\bar{c}_0=\beta_0 l+c_0$ is an approximate value of $\bar{c}$.

4. Errors.

Let R_v, R_y be the errors of v_2 and $y(x)$ when we take (8) and (29) as approximate curves of (7) and (28) respectively, then we have

$$
\left.\begin{aligned}
R_v &= v_2-v_0\\
R_y &= Fx^2+(E+\beta)x+c-(\beta_0x+c_0)
\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\quad(36)
$$

5. When one of v_2, c are known.

When one of v_2, c are known by boring or other method, the computation becomes very simple, that is, we have

$$
\left.\begin{aligned}
&\alpha v_2b\gamma-(\alpha+\beta)=0\\
&\frac{2\gamma c}{\alpha v_2(1-\alpha\beta)}-\frac{2\beta sc}{1-\alpha\beta}+\frac{ac^2}{(1-\alpha\beta)^2}(\alpha^2-\beta^2-2\alpha\beta)=c''
\end{aligned}\right\}\quad\cdots\cdots\quad(37)
$$

and we can solve them with respect to β, and one of v_2, c. Especially when v_2 is known we can easily solve (37) with respect to β, c exactly.

6. When $\beta = 0$.

When $\beta = 0$, that is $b = \bar{b}$, our calculation becomes more simpler, that is $\gamma = 1$, $k = \bar{k} = 1$, $b = 1/v_2$ and (19), (32), (34), (35) become

$$y(x) = \frac{1}{2} a\alpha v_2 x^2 + [a\alpha v_2 c + sv_2 - 1]\alpha x + c \quad \cdots\cdots (19)'$$

$$\left.\begin{aligned} &\frac{1}{2} al^2\alpha v_2 + al\alpha^2 v_2 c + ls\alpha v_2 - l\alpha + c - \bar{c} = 0 \\ &a\alpha^2 c^2 + \frac{2}{\alpha v_2} c - c = 0 \\ &\bar{a}\alpha^2 \bar{c}^2 + \frac{2}{\alpha v_2} \bar{c} - \bar{c}'' = 0 \end{aligned}\right\} \quad \cdots\cdots (32)'$$

$$\left.\begin{aligned} &G_2\delta + L_2\sigma + M_2\bar{\sigma} + N_2 = 0 \\ &G_3\delta + L_3\sigma \cdots\cdots + N_3 = 0 \\ &G_4\delta \cdots\cdots + M_4\bar{\sigma} + N_4 = 0 \end{aligned}\right\} \quad \cdots\cdots (34)'$$

$$\left.\begin{aligned} &G_2 = \frac{alv_1}{\kappa_0}\left[\frac{l}{2}\alpha_0 + (1+2\alpha_0^2)c_0\right] + \frac{lv_1\alpha_0 s}{\kappa_0} - l,\ L_2 = alv_1\alpha_0\kappa_0 + 1 \\ &N_2 = alv_1\kappa_0\left(\frac{l}{2} + \alpha_0 c_0\right) + lv_1\kappa_0 s - (l\alpha_0 + \bar{c}_0 - c_0),\ M_2 = -1 \\ &G_3 = 2\alpha_0 c_0\left(ac_0 - \frac{1}{v_1\kappa_0^3}\right) \quad , \quad L_3 = 2a\alpha_0^2 c_0 + \frac{2}{v_1\kappa_0} \\ &N_3 = a\alpha_0^2 c_0^2 + \frac{2}{v_1\kappa_0} c_0 - c'' \quad , \quad G_4 = 2\alpha_0\bar{c}_0\left(\bar{a}\bar{c}_0 - \frac{1}{v_1\kappa_0^3}\right) \\ &M_4 = 2\bar{a}\alpha_0^2\bar{c}_0 + \frac{2}{v_1\kappa_0} \quad , \quad N_4 = \bar{a}\alpha_0^2\bar{c}_0^2 + \frac{2}{v_1\kappa_0}\bar{c}_0 - \bar{c}'' \end{aligned}\right\} \quad \cdots\cdots (35)'$$

7. Examples.

Example 1.

Following table and $v_1 = 285m/sec$, $l = 70.0m$ are given. Find the value of v_2 and the form of $y(x)$.

$X, \bar{X}$	$10m$	20	30	40	50	60	70
$\varphi(X)$	0.083sec	0.097	0.107	0.118	0.126	0.131	0.137
$\bar{\varphi}(\bar{X})$	0.094sec	0.104	0.116	0.124	0.129	0.135	0.137

Solution:-

Suppose to

$$\varphi(X) = \frac{a}{2} X^2 + sX + c'' \quad \cdots\cdots (i)$$

$$\bar{\varphi}(\bar{X}) = \frac{a}{2} \bar{X}^2 + \bar{s}\bar{X} + \bar{c}'' \quad \cdots\cdots (ii)$$

and substitute given data in these equations, and we get following equations

$$\left.\begin{aligned} 50a + 10s + c'' &= 0.083 \\ 200a + 20s + c'' &= 0.097 \\ 450a + 30s + c'' &= 0.107 \\ 800a + 40s + c'' &= 0.118 \\ 1250a + 50s + c'' &= 0.126 \\ 1800a + 60s + c'' &= 0.131 \\ 2450a + 70s + c'' &= 0.137 \end{aligned}\right\} \quad \cdots\cdots (iii)$$

$$\left.\begin{aligned} 50\bar{a} + 10\bar{s} + \bar{c}'' &= 0.094 \\ 200\bar{a} + 20\bar{s} + \bar{c}'' &= 0.104 \\ 450\bar{a} + 30\bar{s} + \bar{c}'' &= 0.116 \\ 800\bar{a} + 40\bar{s} + \bar{c}'' &= 0.124 \\ 1250\bar{a} + 50\bar{s} + \bar{c}'' &= 0.129 \\ 1800\bar{a} + 60\bar{s} + \bar{c}'' &= 0.135 \\ 2450\bar{a} + 70\bar{s} + \bar{c}'' &= 0.137 \end{aligned}\right\} \quad \cdots\cdots (iv).$$

Deriving normal equations from these, we get

$$\left.\begin{aligned} 1169 \times 10^4 a + 392 \times 10^3 s + 7 \times 10^3 c'' &= 895.05 \\ 392 \times 10^3 a + 14 \times 10^3 s + 280\, c'' &= 34.45 \\ 7 \times 10^3 a + 280\, s + 7\, c'' &= 0.799 \end{aligned}\right\} \quad \cdots\cdots (v)$$

$$\left.\begin{aligned} 1169 \times 10^4 a + 392 \times 10^3 \bar{s} + 7 \times 10^3 \bar{c}'' &= 916.80 \\ 392 \times 10^3 \bar{a} + 14 \times 10^3 \bar{s} + 280\, \bar{c}'' &= 35.60 \\ 7 \times 10^3 \bar{a} + 280\, \bar{s} + 7\, \bar{c}'' &= 0.839 \end{aligned}\right\} \quad \cdots\cdots (vi)$$

and as the roots of these equations we obtain

$$a = -0.000\,016\,91, \quad s = 0.001\,560, \quad c'' = 0.068\,40$$

$$\bar{a} = -0.000\,015\,52, \quad \bar{s} = 0.001\,350, \quad \bar{c}'' = 0.081\,40$$

therefore

$$\varphi(X) = -0.000\,008\,46X^2 + 0.001\,560X + 0.068\,40 \quad \cdots\cdots\cdots\cdots \quad \text{(vii)}$$

$$\bar{\varphi}(\bar{X}) = -0.000\,007\,76\bar{X}^2 + 0.001\,350\bar{X} + 0.081\,40 \quad \cdots\cdots\cdots\cdots \quad \text{(viii)}$$

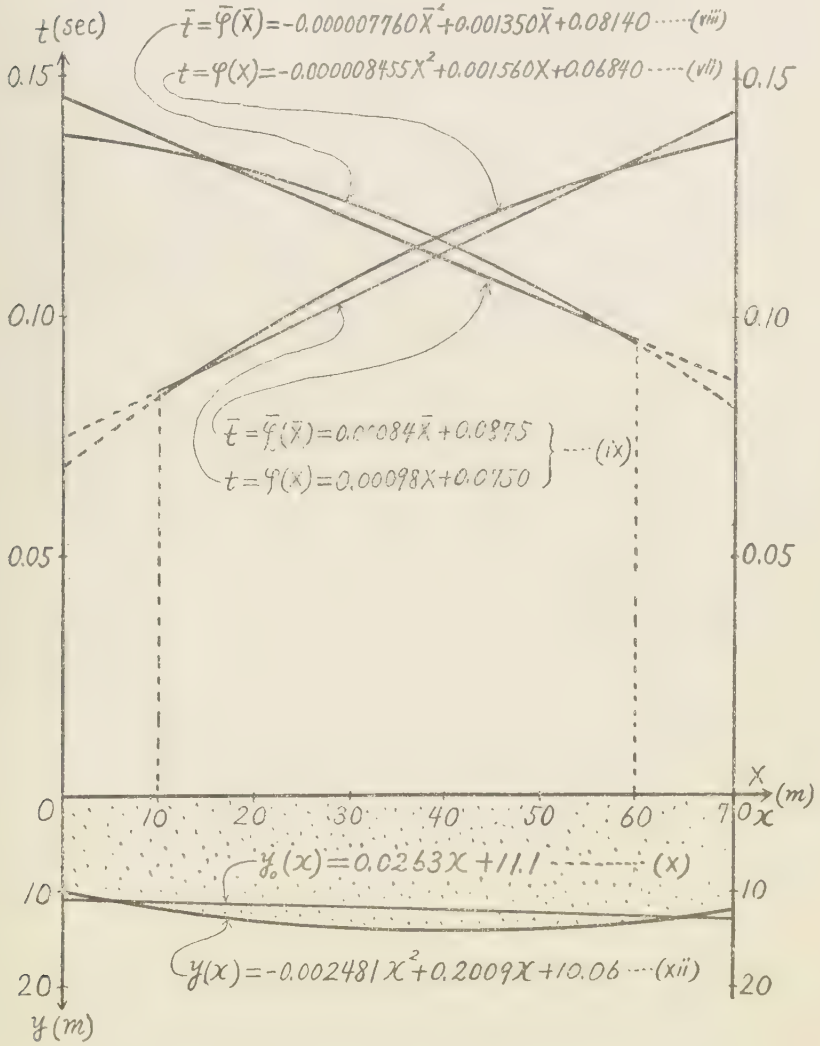

Fig. 3

As shown in Fig. 3, the straight lines which are drawn approximately to (vii) and (viii) are

$$\left.\begin{aligned} t &= \varphi_0(X) = 0.00098X + 0.0750 \\ t &= \bar{\varphi}_0(\bar{X}) = 0.00084\bar{X} + 0.0875 \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad \text{(ix)}$$

and solving $y(x)$ and v_2 from (ix) by ordinary method gives

$$v_0 = 11.0 \times 10^2 m/sec, \quad y_0(x) = 0.0263x + 11.1 \quad \cdots\cdots\cdots\cdots \quad (x).$$

Then we get

$$c_0 = 11.1, \quad \bar{c}_0 = 12.9, \quad V_0 = 10.6 \times 10^2, \quad \alpha_0 = 0.2689, \quad \kappa_0 = 1.0355$$

and $\beta_0 = 0.0263 \fallingdotseq 0$ therefore we take the formulas when $\beta = 0$ and get

$$\begin{aligned} G_2 &= -\ 69.09 \quad , \quad L_2 = 0.9061 \quad , \quad N_2 = -\ 0.5398 \\ G_3 &= -\ 0.01999 \quad , \quad L_3 = 0.006750 \quad , \quad N_3 = 0.006644 \\ G_4 &= -\ 0.02331 \quad , \quad M_4 = 0.006748 \quad , \quad N_4 = 0.005836 \end{aligned}$$

therefore we have following approximate equations

$$\left.\begin{aligned} &-\ 69.09\ \delta + 0.9061\ \sigma \qquad\quad - \bar{\sigma} - 0.5398 = 0 \\ &-\ 0.01999\delta + 0.006750\sigma \cdots\cdots\cdots + 0.006644 = 0 \\ &-\ 0.02331\delta \cdots\cdots\cdots + 0.006748\bar{\sigma} + 0.005836 = 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots \quad (xi).$$

Solving these equations gives

$$\delta = -\ 0.00811, \quad \sigma = -\ 1.008, \quad \sigma = -\ 0.893$$

and we get

$$\alpha = \alpha_0 + \delta = 0.2608, \quad c = c_0 + \sigma = 10.09, \quad \bar{c} = \bar{c}_0 + \bar{\sigma} = 12.01.$$

Recurring this procedure, that is putting 0.2608, 10.09, 12.01 in places of 0.2689, 11.1, 12.9 respectively in above computation, we get

$$\alpha = 0.2464, \quad c = 10.06, \quad \bar{c} = 11.96, \quad v_2 = 1191,$$

and *these values satisfy the equations (32)′ very well.* Therefore we get

$$\begin{aligned} &\alpha[a\alpha v_2 c + s v_2 - 1] = 0.2009 \\ &\frac{1}{2}\, a\alpha v_2 = -\ 0.002481 \end{aligned}$$

and we obtain

$$y(x) = -\ 0.002481x^2 + 0.2009x + 10.06 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (xii).$$

This is our required solution and its graph is shown in Fig. 3.

Example 2.

The following table and $v_1 = 270 m/sec$, $l = 100m$ are given. Find the value of v_2 and the form of $y(x)$.

$X,\ \bar{X}$ (m)	10	20	30	40	50	60	70	80	90	100
$\varphi(X)$ (sec)	0.103	0.128	0.153	0.177	0.200	0.223	0.245	0.266	0.287	0.307
$\bar{\varphi}(\bar{X})$ (sec)			0.366	0.360	0.354	0.346	0.338	0.328	0.318	0.307

Solution:-

Suppose to

$$\varphi(X) = \frac{a}{2}X^2 + sX + c''$$

$$\bar{\varphi}(\bar{X}) = \frac{\bar{a}}{2}\bar{X}^2 + \bar{s}\bar{X} + \bar{c}''$$

and substitute given data and determining the values of a, s, c'', $\bar{a}$, $\bar{s}$, $\bar{c}''$ by the method of least square as in ex. 1, we get

$$a = -0.000\,606\,624, \quad s = -0.002\,627, \quad c'' = 0.07713$$
$$\bar{a} = -0.000\,009\,656, \quad \bar{s} = -0.000\,2126, \quad \bar{c}'' = 0.3763$$

therefore

$$\left.\begin{aligned}\varphi(X) &= -0.000\,003\,315X^2 + 0.002\,627\,X + 0.07713\\ \bar{\varphi}(\bar{X}) &= -0.000\,004\,828\bar{X}^2 - 0.000\,2126\bar{X} + 0.3763\end{aligned}\right\} \cdots\cdots\cdots\cdots \text{(i)}$$

The graphs of travelling-time curves are shown in Fig. 4 and two straight lines drawn approximately to them are

$$\left.\begin{aligned}\varphi_0(X) &= 0.00223X + 0.083\\ \bar{\varphi}_0(\bar{X}) &= -0.00073\bar{X} + 0.385\end{aligned}\right\} \cdots\cdots\cdots\cdots \text{(ii)}.$$

Solving (ii) by ordinary method gives

$$v_0 = 1164, \quad y_0(x) = 0.4616x + 12.69 \cdots\cdots\cdots\cdots \text{(iii)}$$

and

$$\beta_0 = 0.4616, \quad c_0 = 12.69, \quad \bar{c}_0 = \beta_0 l + c_0 = 58.85, \quad V_0 = 1132,$$

therefore

$$\alpha_0 = 0.2385, \quad \gamma_0 = 1.1014, \quad \kappa_0 = 1.0281,$$

and substituting these values in (34), we get

$$\left.\begin{aligned}1.213\delta + 1.057\varepsilon \cdots\cdots\cdots\cdots + 0\\ -1.024\delta + 5.376\varepsilon + 0.0620\sigma - 0.0792\bar{\sigma} - 0.5974 = 0\\ 1.136\delta - 0.7678\varepsilon + 0.4967\sigma \cdots\cdots\cdots + 0.1540 = 0\\ -3.391\delta + 0.9453\varepsilon \cdots\cdots\cdots + 0.0852\bar{\sigma} + 0.3930 = 0\end{aligned}\right\} \cdots\cdots\cdots\cdots \text{(iv)}$$

and the roots of these equations are

$$\delta = -0.0217, \quad \varepsilon = 0.0249$$
$$\sigma = -0.222, \quad \bar{\sigma} = -5.75 .$$

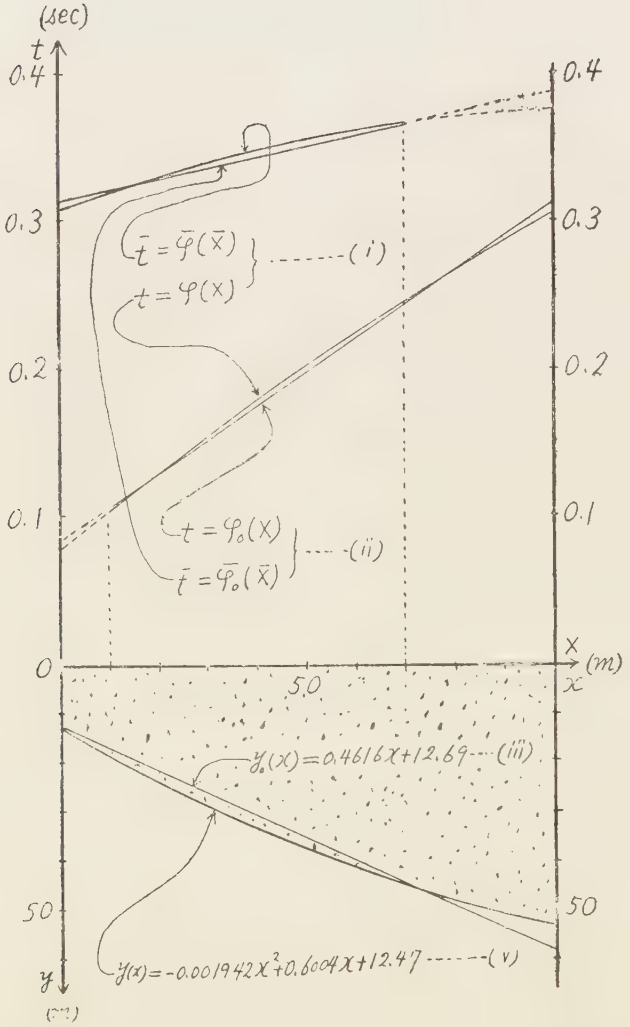

Fig. 4.

Consequently we get

$$\alpha = \alpha_0 + \delta = 0.2168, \quad \beta = \beta_0 + \varepsilon = 0.4865$$
$$c = c_0 + \sigma = 12.47 , \quad \bar{c} = \bar{c}_0 + \sigma = 53.10.$$

Recurring this procedure, secondly we obtain

$$\alpha = 0.2183, \quad \beta = 0.4848$$

$$c = 12.47, \quad \bar{c} = 53.14$$

and *these values satisfy (32) well*, therefore

$$v_2 = v_1 \sqrt{1+\alpha^2}/\alpha = 1266, \quad \gamma = 1.1113$$

and

$$F = -0.001942, \quad E = 0.1156$$

lastly we get

$$y(x) = -0.001942x^2 + 0.6004x + 12.47 \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (v).$$

The graphs of (iii) and (v) are shown in Fig. 4, and their difference is clear at a glance.

8. Some Remarks.

When one of v_2, c is known, in order ot explore $y(x)$, it is sufficient that we know the travelling-time curve (7) only and the travelling-time curve (28) is not necessary.

It is impossible to approximate $\varphi(X)$ with a quadratic formula when $\varphi(X)$ is a complex curve (e. g. when it has some points of inflexion). The method which discussed here is utilizable when $\varphi(X)$ and $\bar{\varphi}(X)$ are both slow curves and can be approximated by a parabola. In the next Journal I will treat of more general cases.

References

(1) See any Book written on the Seismic Prospecting.
(2) R. Adachi: Fundamental Relations on the Seismic Prospectng. Kumamoto Journal of Science Vol. 2, No. 3.

OBSERVATION OF THE TOTAL SOLAR ECLIPSE IN JUNE 20TH, 1955 ON BOARD SHIP. (PRELIMINARY REPORT)

(Recieved February 28, 1956)

Akira Onuki

1. Introduction

It has been said that it is very difficult to make a precise observation of the solar eclipse on board ship, and astronomers have usually selected an isle which lies in the central zone of the eclipse.

However, if the purpose of such observation does not demand the accurate determination of the position of the place, it may be a matter of interest to try such an observation and to know the accuracy we can attain in such a trial. The advantages, if such an observation is proved to be feasible, are obvious. One advantage is that one can freely choose the observation point in the central zone, and another arises from the economical stand point. In fact, by the eclipse in June 20th 1955, our ship had made a run of some 500 seamiles to get rid of the cloudy area, and had successfully caught suitable weather conditions. On the contrary, it can easily be imagined that the rolling of the ship will cause many difficulties and unexpected troubles, so our efforts were used chiefly in eliminating such difficulties. In spite of such effort, our plans were proved to be unsatisfactory in respect to many points. For example, it would be far better if we could go down to the doldrums, where we could expect a more calm sea. Even though the results of this expedition were not fully satisfactory and some of our plans were found to fail utterly, we feel there is some hope in the near future if the purposes of observation are limited, and provided it is carefully planned and the party is equipped with appropriate instruments.

2. The expedition.

The expedition was planned in the beginning of 1955 and the request for the use of the "Keiten-Maru", 260 tons, the training ship belonging to Kagoshima University, was kindly granted by the University authorities. The members of the expedition were, together with 37 crew members of the ship, as follows.

As. Prof.	A. Onuki	Kumamoto University
As.	T. Chikazawa	〃
As.	K. Kaminishi	〃
	S. Izumi	Photographer
As. Prof.	C. Fuzita	Kagoshima University

The captain, T. Genga, As. Prof. of Kagoshima University, also engaged in taking records

of meteological and oceanographical data throughout the eclipse time.

The "Keiten-Maru" departed from Kagoshima-port on 11th June, and arrived on 17th at the point N 11° E 135°, which is nearly in the center of the total eclipse zone. The weather was not so fine and gradually became worse, so we determined to change our position 500 sea miles south eastward along the central zone. According to radio information from Manila it was considered that we would have a much better chance of success if we made such a change of position. Early in the morning of the eclipse day our ship stopped at the point N 9°16′ E 134° 31′. The sky was becoming clear though there were specks of cloud remaining here and there. The sea surface was calm and waves of one or two meters in height rolled the ship slightly.

The predicted times of the first and second contact were 12h 30m 5s and 13h 58m 54s (Japan Standard Time). Both of these were 21 seconds earlier than the really observed times. Sextant observation was carefully repeated just before the contact, but between the first and the second contact the ship had twice to change position at full speed to evade squalis, and consequently, such discrepancy of predicted and observed time was considered to be inevitable. The position of the ship at the time of second contact determined from the calculation is shown in Fig. 1.

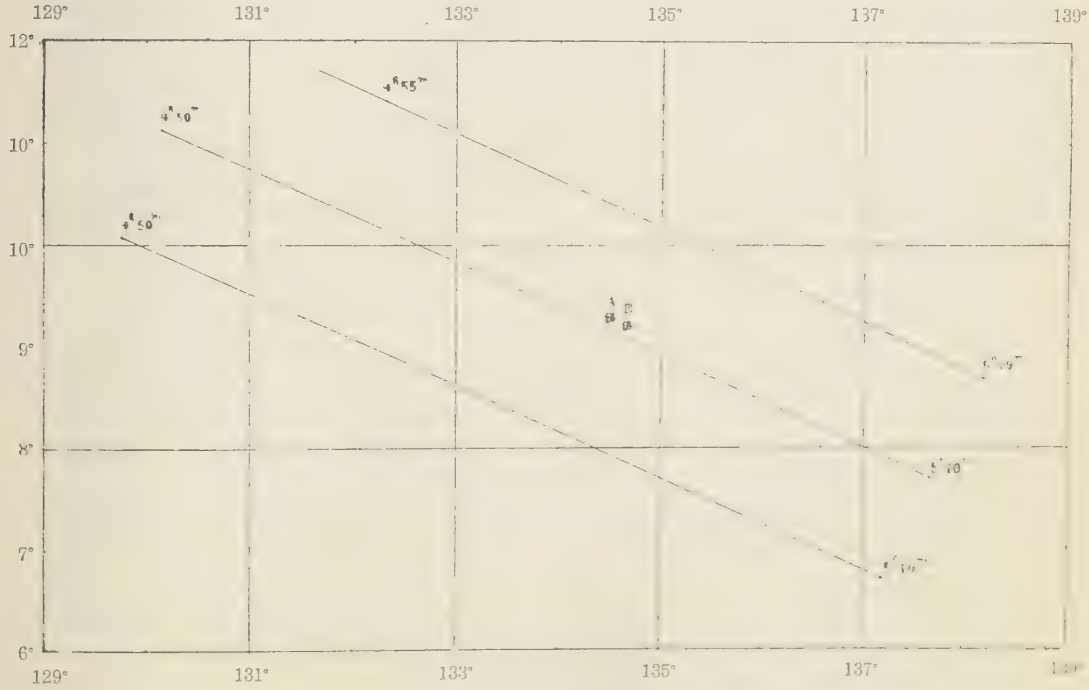

Fig. 1 A ; The position from the sextant observation.
B ; The calculated position.

3. Plans of observation

As already stated, this experiment is utterly hopeless for any observation which requires the accurate geographical position of the ship. Naturally our plans of observation were confined to those which chiefly concern the intensive properties of the sun beam. For these purposes, a camera of focal length 4.5 cm with a lense diameter 1.4 cm attached with hemispherical reflecting mirror was used so as to catch the whole sky in one film. The object sought in using this camera was to determine the relative light intensity of various parts of the sky.

At the sametime, other simple mechanical devices were used to measure the light intensity of the selected spot near the sun. (Angular distance from the sun reaches to about 30°). This was composed of 170 cm long square cylinder having a window that guides the sun beam directly to a photo-electric tube or photocell, which can be replaced by a camera of focal length 15 cm if desired. Between the first and fourth contacts, 8 times of measurements, each of which included 3 different readings, namely with photo-tube, with photocell, and with photographic plate, were planned. It was determined that the whole-sky camera would take 40 pictures throughout the eclipse time.

Besides, another 27 cm focal length camera was prepared to photograph the corona directly. This camera was to be used for determining the maximum exposure time allowed under such conditions. In fact, various times of exposure between 1/100 sec. and 1 sec. were tested at the time of the eclipse. From the results obtained, we have concluded that the maxrmum time-limit is approximately 1/25 second under such circumstances, unless special equipment were employed for stabilizing the camera. Of course this time length of exposure is not sufficient for the outer region of the corona even for a short focal length camera. But one may hope that the recent progress of the highly sensitized film together with the use of stabilizing device will in some degree remove this difficulty in the near future.

4. The eclipse.

The sky around the sun was perfectly clear during the total eclipse, its duration being 6 m. 10s, and we could observe the beautiful corona streaming out from the sun's surface towards the equatorial plane of the sun, extending for a distance approximately equal to three times the diameter of the sun. Large pink flames of prominence erupting from the limb of the sun were also seen even by the naked eye.

"The shadow bands" were clearly observed by the several members of the crew who had been previously distributed on the decks for purpose of observing closely the movement of these bands.

The previously determined whole plan of the observation was carried out throughout the time of eclipse except that a little trouble occured when a sudden shock caused by a wave tumbled the telescopic photometer and broke a vacuum tube.

The observed time of 4th contact is 15h 20m 51s. The films taken were carefully

placed in the icebox in the store-room of the ship, where the temperature was kept exactly at −2°C, so as to protect them from the warm moisture of the southern sea.

The ship returned to Kagoshima on 19th July after two weeks fishing along the north coast of New Guinea.

5. The Results.

The chief results of observations obtained in this expedition are summarised in the following tables except those of the biological and some of the oceanographical. The meaning of the figures and necessary remarks will be found under these tables.

Table 1.

Observed Values by the Photoelectric-tube and Photocell.

t	photo-tube	t	photocell
12h 20m	91	12h 25m	38
12 50	69	12 55	33
—	—	13 20	29
—	—	13 45	20
—	—	14 00	0.2
—	—	14 15	11
—	—	14 40	54
—	—	15 19	105

t denotes Japan Standard Time. Each value in the second and fourth column is the arithmetic mean of the four values at the selected spots around the sun. (The angular distances from the sun are about 7° and 20° in the case of photo-tube and of photocell respectively.)

Table 2.

Observed data by the photocell at 13h 30m. (Japan Standard Time).

Altitude	Obs. data
about 54°	4 8
〃 45	3 7
〃 36	3 0
〃 27	3 3
〃 18	2 3
〃 9	1 6

The altitude of measured spot was roughly measured along the vertical circle on which the sun was situated at that time.

The plans stated in § 4 were carried out as follows. The sky was photographed in each 5 minutes from 12h 35m to 14h 45m by the whole sky camera, time of each exposures being 1/10 sec, and f = 8. A circular disc of diameter 30 cm was fitted up in front of camera so as to shelter the direct ray of the sun during the partial eclipse. Konipan U. S. S. Films were used.

Of 36 films thus taken, 2 failed by the unexpected roll of the ship, and remaining 34 were found to be available for the estimation of the light intensity of the sky. Some of them are shown in Fig. 2, 3, 4, 5 and 6. As it was considered from these films that the relative blackening of special areas on the film will well express the whole trend of light intensity of the various parts of the sky, areas of 11° …… 17° and 73° …… 84° in altitude are selected and measured photometrically. These figures are shown in the table together with the blackening of central part of films.

Table 3. Blackening of the films of the whole sky camera

Film Nos.	t			z	180°—A	x	Blackening		
	h	m	s				o	a	g
1	12	32		16°.5	29°.5	0.02	104	204	200
2		34		16.7	30.9	0.04	112	209	199
3		55		18.8	39.4	0.28	100	201	188
4	13	02	20	20.6	44.6	0.36	99	204	162
5		07	38	21.5	46.7	0.42	97	178	171
6		13		23.0	49.8	0.48	102	154	180
7		23		25.0	52.8	0.60	82	150	170
9		27		25.8	53.9	0.65	85	158	144
10		32		26.8	55.2	0.70	70	120	133
11		38		28.0	56.5	0.77	70	119	149
12		43		29.0	57.6	0.83	58	105	119
13		48		30.1	58.5	0.89	47	96	118
14		52		30.9	59.2	0.93	36	78	90
15	14	00	30	32.8	60.6	1.00	12	11	13
16		02		33.1	60.8	1.00	12	11	13
17		05		33.7	61.2	1.00	12	12	17
18		10		34.8	61.9	0.94	41	74	76
20		15		35.9	62.5	0.88	50	102	126
21		16	32	36.2	62.7	0.86	49	97	99
22		19		36.8	62.9	0.83	60	82	96
23		21		37.2	63.2	0.81	69	116	124
24		25		38.1	63.6	0.76	78	149	136
25		30		39.2	64.0	0.70	76	157	183
26		35		40.3	64.4	0.64	86	162	208
27		41		41.6	65.0	0.58	99	162	213
28		45		42.7	65.3	0.53	89	160	219
29		50		43.7	65.6	0.47	100	206	226
30		54	26	44.7	65.9	0.42	114	199	254
31		59		45.7	66.2	0.37	111	198	245
32	15	05		47.1	66.5	0.30	100	191	265
33		10		47.5	66.6	0.24	92	216	278
34		14	25	48.5	66.8	0.19	100	195	275
35		20		50.4	67.2	0.12	125	215	291
36		25		51.6	47.2	0.06	121	206	239

t: Japan Standard Time.
z and A: zenith distance and azimuth of the sun respectively.
x: degree of obscuration.
o: the blackening of the central part of the film.
a: the blackening of the annular area between 11° ····· 21° in the zenith distance.
g: the blackening of the annular area between 73° ····· 84° in the zenith distance.

Table 4. Temperature of the sea water

t (h m) \ depth	0~20cm	50m
	°C	°C
12 15	29.5	28.35
34	29.9	28.35
45	29.45	28.22
13 03	29.63	28.62
31	29.16	28.50
58	28.84	28.50
14. 05	28.85	28.55
10	28.84	28.62
20	28.81	28.22
34	29.01	28.63
50	29.08	28.56
15 04	29.45	28.58
22	29.25	28.57
34	29.47	28.40

Fig. 3 Picture with whole sky camera(Film No. 14)

Fig. 2 Picture with whole sky camera, (Film No. 9)

Fig. 4 Picture with whole sky camera, (Film No. 17)

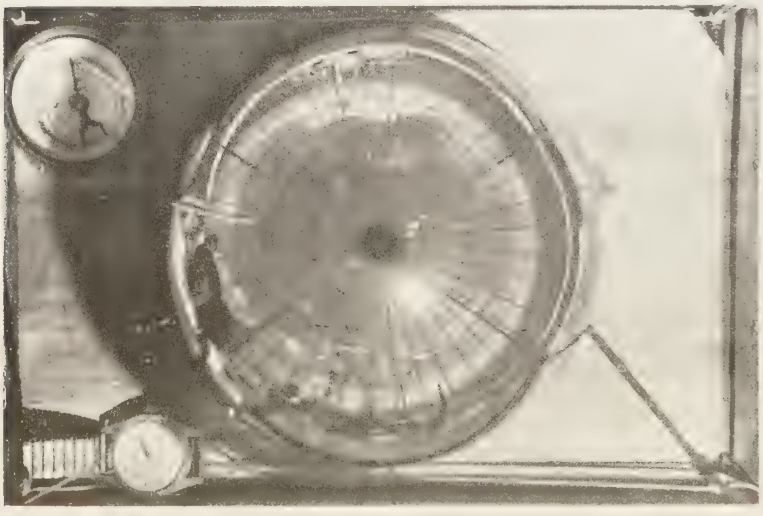

Fig. 5 Picture with whole sky camera, (Film No. 18)

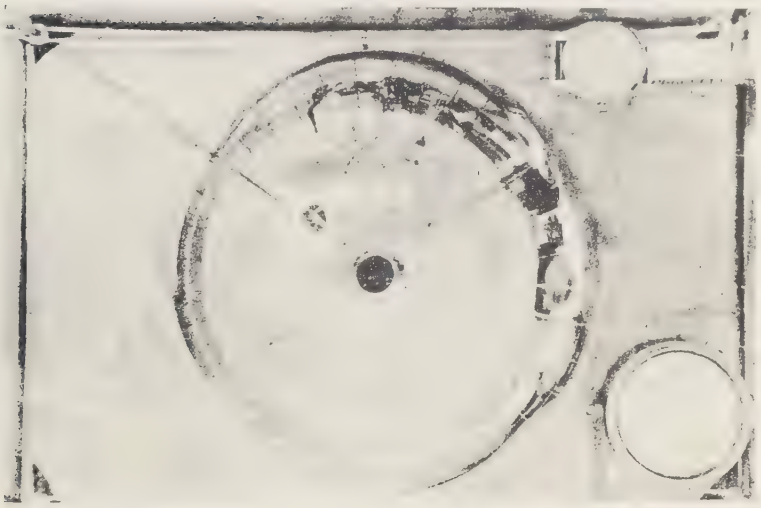

Fig. 6 Picture with whole sky camera, (FilmNo. 24)

6. The temperature of the sea water was measured by the "reversing thermometer" in various depths. It is interesting that the cooling down of the sea water near the surface at the total eclipse was nearly the same amount of the atmospheric temperature drop,

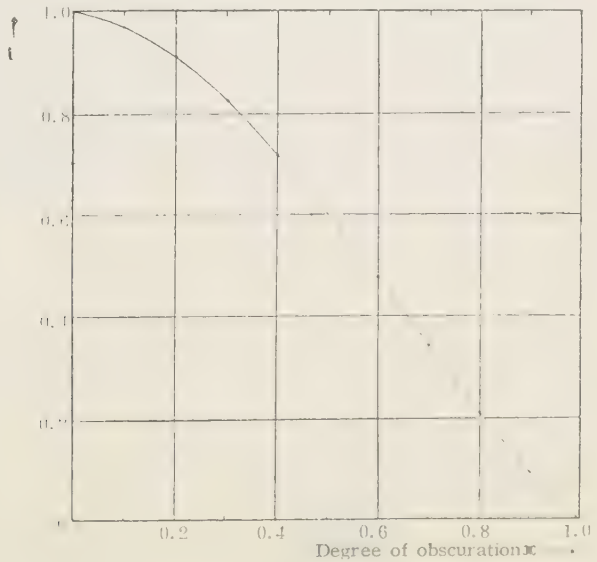

Fig. 7

namely about one degree. In considering the large heat capacity of the water, this drop is much bigger than was previously expected. The most believable reasoning for it is that the drop in the shallow water is probably governed chiefly by the decreasing of the direct solar radiation going through without heating the surrounding water. As the ship was slowly drifted by the oceanic current and the wind throughout the eclipse time, the effects of the wind and and that of the fluctuation from the convection of sea water might be considerablly great, it may be too much to say that the only reason for the cooling down is the decreasing of the sun's radiation. Figure 7. shows how the total amount of the solar radiation decreases with the degree of obscuration throughout the time of the eclipse. The curve shown does not include the intensity of scattered light by the air particle, and consequently, the constant amount should be added L to to get the observed total intensity. The formulae used are

$$\frac{L}{L_0} = 1 - \int_{1-2x}^{1} \frac{1}{\pi} \cos^{-1} Y(\eta, x)\, w(\eta)\, \eta d\eta,$$

when $0 \leqq x \leqq \frac{1}{2}$,

and $$\frac{L}{L_0} = \int_{2x-1}^{1} \left\{1 - \frac{1}{\pi} \cos^{-1} Y(\eta, x)\right\} w(\eta)\, \eta d\eta,$$

when $\frac{1}{2} < x \leqq 1$,

where $Y(\eta, x) = \dfrac{\eta^2 + 4(1-x)^2 - 1}{4\eta(1-x)}$,

and $$w(\eta) = \eta \left\{1 + \frac{3}{2}\sqrt{1-\eta^2}\right\},$$

x being the degree of obscuration.

7. Shadow band.

About twelve men who were distributed to watch the shadow band observed the "shadow band-like" phenomena immediately before the total eclipse began. Their statements about these agree in many points, and are summarised here as follows.

1. The width of the stripes was about 2~5 cm, moving with the velocity of one or two meters per second toward the sun (westward, see Fig. 8). They were clearly observed on the deck of the ship, but were not seen on the surface of the sea.

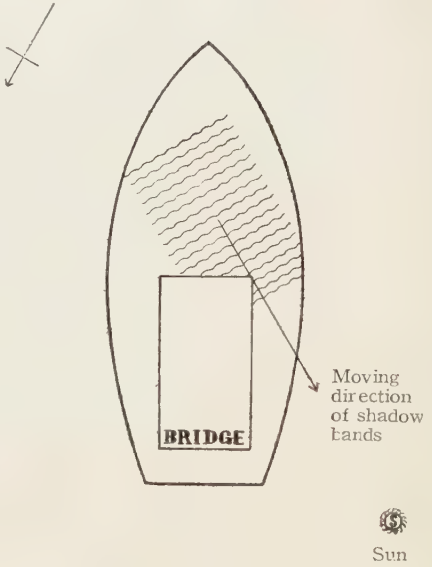

Fig. 8

2. The shadow bands were seen also at the end of the total eclipse, and the figure and the moving direction were the same as before, that is they moved toward the sun.
3. There was a light breeze from the southwest, and no specks of cloud or mist were seen near the sun. The whole thing appeared and disappeared like a phantom, its duration being one minute or two.

8. Conclusion.

The other meteorological and oceanographical data in detail obtained by this expedition will be published later (in Japanese) in cooperation with Kagoshima University. The detailed investigations are now in progress, and here we want to make a preliminary report about the possibility of on board ship observation only.

Our final conclusions are as follows; It cannot be said that it is utterly hopeless to observe the total eclipse on board ship without landing on an isle. It is desirable that the ship used for these purposes be as large as possible, having some devices for the elimination of the effects from the roll of the waves. In case the central zone of the eclipse passes the equator of the earth, the waves in this area are practically harmless for the short exposure time under 1/10 second, particularly in fine summer weather. If suitable equipment were available, we think it might not be impossible to prolong the exposure time till about one second.

Then, using a large aperture lense and high speed sensitive film, much more of the outer region of the corona may be photographed easily. But, when one uses the long focal length camera, a difficulty arises in determining the posilion of the area photographed relative to the sun, or relative to the azimuth-altitude, because no terrestrial object can be available in the midst of the ocean and the direction of the ship is continually changing.

Such difficulties clearly limit greatly the type of observation possible. However, while acknowledging these limitations, we still believe that many new experiments can be carried out from on board ship, which will lead to fuller knowledge of the solar eclipse.

9. Acknowledgement.

The author wishes to express his sincere thanks to Prof. S. T. Nakamura and to Prof. Yamamoto for their kind guidance and encouragement throughout this expedition, and also to the Governor of Kumamoto prefecture, to the Mayor of Kumamoto City, to the Bank Association of Kumamoto, and to the other people who kindly contributed to the fund for this expedition.

Particularly we wish to express our gratitude to Mr. Goto Zezan and to Mr. Y, Otani, who gave as continual encouragement and much helpful advice.

We are greatly indebted also to the Shin Nippon Chisso Company for their kindness in permitting the use of their instruments, and to Kagoshima University for their courtesy in placing their ship, "the Keiten Maru" at our disposal.

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